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ON SEMIALGEBRAIC TRANSFORMATION GROUPS I

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ABSTRACT. We survey some recent developments in semialgebraic transformation group theory.

CONTENTS

INTRODUCTION

In this article we survey some recent developments in semialgebraic transformation group theory. In particular, we discuss topological properties of semialgebraic sets with semialgebraic actions of semialgebraic groups. This concept is an semialgebraic analogue of topological or smooth actions of Lie groups. For topological or smooth actions, see [3], [12] or [21].

The actual development of the theory of semialgebraic sets, as well as the theory of semi-analytic sets, began with the work [25] of S. Łojasiewicz in 1964. In the seventies G. W. Brumfiel had already laid down a program for what we call "semialgebraic topology" in [4]. Around 1979 two different approaches to semialgebraic topology emerged independently. One is the "abstract" approach due to M. Coste and M.-F. Roy [7, 8, 40], and the other is "geometric" approach due to H. Delfs and M. Knebusch [9, 10, 11].

Semialgebraic category lies between toplogical category and algebraic category, and it is used to understand the geometric structure of an algebraic variety. Semialgebraic category is less rigid than algebraic category, and in some sense similar to PL (piece-wise linear) category in topology. We have the following relations among several categories in which topologists are interested:

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$$\begin{array}{ccccc} & & \text{Definable} & & \\ & & \cup & & \\ \text{Algebraic} & \subset & \boxed{\text{Semialgebraic}} & \subset & \text{Topological} \\ \cup & & \cup & & \cup \\ \text{Nonsingular algebraic} & \subset & \text{Nash} & \subset & \text{Smooth} \\ & & \cap & & \\ & & \text{Analytic} & & \end{array}$$

Nash manifolds and Nash maps are defined in Section 2. Throughout this article, all Nash manifolds are affine.

A central question in the semialgebraic category is whether a given topological situation be semialgebraically realized. Another question is to compare the semialgebraic objects with the corresponding objects in the other categories.

In this article we only consider the semialgebraic sets in $\mathbb{R}^n$ equipped with the subspace topology induced by the usual topology of $\mathbb{R}^n$. The detailed arguments of this article will be given in the forthcoming papers to be published elsewhere.

Here we give some basic references for the readers which are related to our topics.

Category	Nonequivariant	Equivariant
Semialgebraic	[2, 9, 10, 11]	[6, 33, 34, 35]
Nash	[2, 43]	[22, 23]
Real Algebraic	[2, 1, 20]	[13, 14, 15]
Definable	[16, 31]	[24, 38]
Transformation group theory		[3, 12, 21]

1. Semialgebraic sets and maps

In this section we review some background materials from semialgebraic category.

An (real) **algebraic variety** in $\mathbb{R}^n$ is the set of common zeros of finitely many polynomials $p_1, \ldots, p_k \colon \mathbb{R}^n \to \mathbb{R}$. Note that each algebraic variety can be given by the zero set of a single polynomial $p = \sum_{i=1}^{k} p_i^2$. We are interested in the properties of a larger class of subsets of $\mathbb{R}^n$ than that of algebraic varieties, namely the semialgebraic sets.

The class of **semialgebraic sets** in $\mathbb{R}^n$ is the smallest collection of subsets containing all $\{x \in \mathbb{R}^n \mid p(x) > 0\}$ for each polynomial $p \colon \mathbb{R}^n \to \mathbb{R}$ which is stable under finite union, finite intersection and complement.

It follows from the definition of a semialgebraic set that a subset M of $\mathbb{R}^n$ is semialgebraic if and only if there exist polynomials f_{ij} and g_{ij} for $i = 1, \ldots, p$ and $j = 1, \ldots, q$, such that

$$M = \bigcup_{i=1}^{p} \{x \in \mathbb{R}^n \mid f_{ij}(x) > 0,\ g_{ij}(x) = 0 \ \text{ for all } \ j\}.$$

It is easy to see that finite unions and finite intersections of semialgebraic sets are semialgebraic and that $\mathbb{R}^n$ and $\varnothing$ are semialgebraic sets in $\mathbb{R}^n$. Moreover, let M and N be semialgebraic sets in $\mathbb{R}^n$, then $M - N$ is also semialgebraic in $\mathbb{R}^n$. The cartesian product of two semialgebraic sets is semialgebraic.

The following proposition is one of the basic properties of semialgebraic sets.

Proposition 1.1 (Tarski-Seidenberg principle [42, 2])**.** *Let A be a semialgebraic set in $\mathbb{R}^m$ and $f\colon \mathbb{R}^m \to \mathbb{R}^n$ a polynomial map. Then the image $f(A)$ is also a semialgebraic set in $\mathbb{R}^n$.*

The following is the pleasant consequence of the Tarski-Seidenberg principle.

Proposition 1.2. *The closure and the interior of a semialgebraic set are semialgebraic.*

We now define semialgebraic maps. Let M and N be semialgebraic subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. A map $f\colon M \to N$ is said to be **semialgebraic** if it is continuous and its graph $\mathrm{Gr}(f)$ is a semialgebraic set in $\mathbb{R}^m \times \mathbb{R}^n$.

It is easy to show that the composition of two semialgebraic maps is semialgebraic and that the inverse of a semialgebraic homeomorphism is also semialgebraic. The following result is an important property in semialgebraic geometry.

Proposition 1.3. *Let M, N be semialgebraic sets and $f\colon M \to N$ a semialgebraic map.*

(1) *If A is a semialgebraic subset of M, then its image $f(A)$ is semialgebraic.*
(2) *If B is a semialgebraic subset of N, then its preimage $f^{-1}(B)$ is semialgebraic.*

A smooth submanifold of $\mathbb{R}^n$ is called a (affine) **Nash manifold** in $\mathbb{R}^n$ if it is a semialgebraic set in $\mathbb{R}^n$. Some times we call a Nash manifold in $\mathbb{R}^n$ by a **Nash submanifold** of $\mathbb{R}^n$. A map between two Nash manifolds is called **Nash** if it is a semialgebraic map of the class $\mathcal{C}^\infty$. Note that a Nash manifold and a Nash map are automatically of the class $\mathcal{C}^\omega$ (analytic) by a theorem of B. Malgrange [28] (see [2, Proposition 8.1.8]).

S. Łojasiewicz [25] and H. Hironaka [18] proved the following semialgebraic triangulation theorem.

Proposition 1.4. *Given a finite system of semialgebraic sets M_i in $\mathbb{R}^n$, there exists a finite open simplicial complex K in $\mathbb{R}^{n+1}$ and a semialgebraic homeomorphism $\tau\colon |K| \to \bigcup_i M_i$ such that*

(1) *each M_i is a finite union of some of the $\tau(\sigma)$, where σ is an open simplex of K.*
(2) *$\tau(\sigma)$ is a Nash submanifold of $\mathbb{R}^n$ and τ induces a Nash diffeomorphism, for every open simplex σ in K.*

Here, the finite open simplicial complex means that the finite union of some open simplices of some finite simplicial complex.

Proposition 1.4 implies that every semialgebraic set has a finite number of connected components. Moreover, every connected component of a semialgebraic set is also semialgebraic.

Remark 1.5. A **semialgebraic space** is an object obtained by pasting finitely many semialgebraic sets together along open semialgebraic subsets. R. Robson [39] proves that every 'regular' semialgebraic space admits a semialgebraic embedding into $\mathbb{R}^n$ for some n. In other words, every regular semialgebraic space is semialgebraically homeomorphic to a semialgebraic set in $\mathbb{R}^n$ for some n.

2. Semialgebraic groups

In this section we define the semialgebraic groups and give some basic properties of semialgebraic groups.

A semialgebraic set G ($\subset \mathbb{R}^n$) is called a **semialgebraic group** if it is a topological group whose multiplication and inversion

$$\begin{array}{ll} \mu & : G \times G \to G, \quad (g,h) \mapsto gh \\ i & : G \to G, \quad g \mapsto g^{-1} \end{array}$$

are semialgebraic maps.

Analogously, we also define an affine Nash group.

The following is the semialgebraic analogue of the corresponding fact in Lie group theory.

Proposition 2.1. *Let G be a semialgebraic group.*

(1) *The closure of any semialgebraic subgroup of G is a closed semialgebraic subgroup of G.*
(2) *The center $Z(G)$ is a closed normal subgroup of G.*
(3) *The connected component G_0 of G containing the identity e is a closed normal semialgebraic subgroup of G.*
(4) *The normalizer $N(H)$ of a closed semialgebraic subgroup H of G is a closed semialgebraic subgroup of G.*

As an immediate consequence of Proposition 1.3, we have the following.

Proposition 2.2. *Let G, G' be semialgebraic groups and $f\colon G \to G'$ a semialgebraic homomorphism.*

(1) *If H is a semialgebraic subgroup of G, then $f(H)$ is a semialgebraic subgroup of G'.*
(2) *If K is a semialgebraic subgroup of G', then $f^{-1}(K)$ is a semialgebraic subgroup of G.*
(3) *In particular,* $\mathrm{Ker}(f)$ *is a normal semialgebraic subgroup of G.*

Moreover, we have the following implications on groups:

$$\begin{array}{ccccc} \text{an algebraic group} & \Rightarrow & \text{a Nash group} & \Rightarrow & \text{a Lie group} \\ \uparrow & & \Downarrow\uparrow & & \\ \text{a compact Lie group} & \to & \text{a semialgebraic group} & & \end{array}$$

where $A \to B$ means that A admits B structure.

We now discuss some basic properties of semialgebraic linear groups. A semialgebraic group is called a **semialgebraic linear group** if it is semialgebraically isomorphic to a semialgebraic subgroup of some general linear group $\mathrm{GL}_n(\mathbb{R})$. Note that $\mathrm{GL}_n(\mathbb{R})$ is a semialgebraic set in $\mathrm{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. We remark that there exists a compact semialgebraic group which is not semialgebraic linear, see [37]. Every compact subgroup of a semialgebraic linear group is an algebraic subgroup, hence it is also a semialgebraic linear group.

It is well known that for a closed subgroup H of a compact Lie group G there exists an orthogonal representation $\rho\colon G \to \mathrm{O}(n)$ and a point $v \in \mathbb{R}^n(\rho)$ such that $G_v = H$. We have the semialgebraic analogue of this fact as follows.

Theorem 2.3 ([37])**.** *Let G be a compact semialgebraic linear group and H a closed subgroup of G. Then there exists a semialgebraic faithful representation $\rho\colon G \to \mathrm{O}(n)$ for some n, and a point $v \in \mathbb{R}^n(\rho)$ such that $G_v = H$.*

3. Semialgebraic G-sets and semialgebraic G-maps

In this section we study semialgebraic actions of semialgebraic groups on semialgebraic sets. For general terminology and theory of transformation groups we refer the reader to [3, 12, 21].

By a **semialgebraic transformation group** we mean a triple (G, M, θ), where G is a semialgebraic group, M is a semialgebraic set, and $\theta\colon G \times M \to M$ is a semialgebraic map such that;

(1) $\theta(g, \theta(h, x)) = \theta(gh, x)$ for all g, $h \in G$ and $x \in M$,
(2) $\theta(e, x) = x$ for all $x \in M$, where e is the identity of G.

The map θ is called a **semialgebraic action** of G on M. The semialgebraic set M, together with a given action θ, is called a **semialgebraic G-set**. We briefly denote $\theta(g, x)$ by gx.

A map $f\colon M \to N$ between semialgebraic sets is said to be a **semialgebraic G-map** if

(1) f is a continuous G-map, that is, $f(gx) = gf(x)$ for all $g \in G$ and $x \in M$,
(2) f is a semialgebraic map between ordinary semialgebraic sets.

Analogously, we can define the Nash G-manifold and the Nash G-map, where G is a Nash group.

Now let us list some of the fundamental properties in the theory of semialgebraic transformation groups.

Proposition 3.1 ([5])**.** *Let G be a compact semialgebraic group and M a semialgebraic G-set. Then the orbit space M/G exists as a semialgebraic set such that the orbit map $\pi\colon M \to M/G$ is semialgebraic.*

This proposition implies the following equivariant semialgebraic Urysohn's lemma of semialgebraic G-sets.

Proposition 3.2 ([35])**.** *Let G be a compact semialgebraic group and M a semialgebraic G-set, and let A and B be disjoint closed semialgebraic G-subsets of M. Then there exists G-invariant semialgebraic map $f\colon M \to [0, 1]$ such that $f^{-1}(0) = A$, $f^{-1}(1) = B$.*

Theorem 3.3 ([37])**.** *Let G be a compact semialgebraic group. Then every semialgebraic G-set has only finitely many orbit types.*

Let G be a compact semialgebraic group. For a semialgebraic G-set M and a point $x \in M$, then the isotropy subgroup $G_x = \{g \in G \mid gx = x\}$ is a closed semialgebraic subgroup of G. Proposition 3.1 implies that the homogeneous space G/G_x is also a semialgebraic G-set. On the other hand, the orbit $G(x)$ is a semialgebraic G-subset of M because $G(x) = \theta(G \times \{x\})$. As in the theory of Lie group actions we have that the natural map

$$\alpha_x\colon G/G_x \to G(x), \quad gG_x \mapsto gx$$

is a semialgebraic G-homeomorphism. Moreover, the fixed point set $M^G = \{x \in M \mid gx = x \text{ for all } g \in G\}$ is a closed semialgebraic subset of M even if G is not compact. Furthermore, we have the following.

Proposition 3.4 ([35]). *Let G be a compact semialgebraic group and M a semialgebraic G-set. For any subgroup H of G, $M_{(H)} = \{x \in M \mid G_x = gHg^{-1} \text{ for some } g \in G\}$ is a semialgebraic G-subset of M.*

In particular, if H is not a closed subgroup of G, $M_{(H)} = \varnothing$.

4. G-CW complex structure and embedding theorem

In this section we discuss G-CW complex structures and semialgebraic G-embeddings of semialgebraic G-sets. We first consider the equivariant version of the local triviality theorem of semialgebraic maps as follows.

Proposition 4.1 ([37]). *Let G be a compact semialgebraic group, M a semialgebraic G-set and $\pi\colon M \to M/G$ the semialgebraic orbit map. Then there exists a finite decomposition $\{B_i\}$ of M/G into semialgebraic G-subsets such that for each B_i there exists a semialgebraic G-homeomorphism $\varphi_i\colon \pi^{-1}(B_i) \to B_i \times \pi^{-1}(b_i)$ such that $\pi|_{\pi^{-1}(B_i)} = p_i \circ \varphi_i$, where $b_i \in B_i$ and $p_i\colon B_i \times \pi^{-1}(b_i) \to B_i$ is the projection.*

$$\begin{array}{ccc} \pi^{-1}(B_i) & \xrightarrow{\varphi_i} & B_i \times \pi^{-1}(b_i) \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle p_i} & \\ & B_i & \end{array}$$

In the nonequivariant case, the local semialgebraic triviality theorem of semialgebraic map was proved by R. M. Hardt [17] (see, [2]).

As an application of Proposition 4.1, we can construct a G-CW complex structure on a semialgebraic G-set as follows: We first take a finite decomposition $\{B_i\}$ of M/G into semialgebraic subsets as in Proposition 4.1. By Proposition 1.4, there exist a finite open simplicial complex K and a semialgebraic homeomorphism $\chi\colon K \to M/G$ which is compatible with $\{B_i\} \cup \{M_{(H)}\}$. Then $\{\pi^{-1}(\chi(\sigma)) \mid \sigma \text{ is an open simplex of } K\}$ is a desired finite open G-CW complex structure of M. Thus we have

Theorem 4.2 ([33, 35, 37]). *Let G be a compact semialgebraic group. Let M be a semialgebraic G-set and N a semialgebraic G-subset of M. Then there exists a pair (X, A) of finite open G-CW complexes such that*

1. *the underlying spaces X and A are equal to M and N, respectively,*
2. *each open G-cell c of X is a semialgebraic G-set, and hence its closure $\overline{c}$ is also a semialgebraic G-set, and*
3. *each characteristic G-map is semialgebraic.*

In particular, if G is finite, then we can take X to be a finite open simplicial G-complex. Furthermore, if M is compact, we can take X to be a complete finite G-CW complex. We call such a G-CW complex X, as in Theorem 4.2, a **semialgebraic G-CW complex structure** on M.

A semialgebraic G-set is called **affine** if it is semialgebraically G-homeomorphic to a G-invariant semialgebraic set in some semialgebraic representation space of G. The following is the equivariant semialgebraic embedding theorem.

Theorem 4.3 ([37]). *Let G be a compact semialgebraic linear group. Then every semialgebraic G-set can be equivariantly and semialgebraically embedded in some semialgebraic orthogonal representation space of G. In other words, any semialgebraic G-set is an affine semialgebraic G-set.*

It is obtained from Theorems 2.3 and 4.2. We remark that there exist a compact semialgebraic group G which is not semialgebraic linear and a compact semialgebraic G-set M with one orbit type which is not affine, see [37]. This explains why we only consider semialgebraic linear groups in Theorem 4.3.

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