Trends in Mathematics
Information Center for Mathematical Sciences
Volume 5, Number 1, June 2002, Pages 53–58

p-LOCAL STABLE SPLITTING OF THE COMPLEX STIEFEL MANIFOLD

YEONG-MEE YANG

ABSTRACT. H.Miller introduced a filtration of the complex Stiefel manifold which splits stably and proved that the starata of the filtration are vector bundles, so that Stiefel manifolds are stably equivalent to a wedge of the corresponding Thom spaces. In this paper, we shall give a finer p-local stable splitting of the complex Stiefel manifolds by using the Adams operations.

1. INTRODUCTION

Let $V_{n+m,n}$ be the Stiefel manifold of orthonormal n-frames in $\mathbb{C}^{n+m}$. We regard this $V_{n+m,n} \subset \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{n+m})$ as the space of isometric linear maps from $\mathbb{C}^n$ to $\mathbb{C}^{n+m}$. Then we can represent each element of $V_{n+m,n}$ as a pair (g, h) where $g \in \mathrm{End}(\mathbb{C}^n), h \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$ satisfying $g^*g + h^*h = 1_n$. H.Miller [2] has introduced a filtration $\{(1,0)\} = F^0 \subset F^1 \subset \cdots \subset F^n = V_{n+m,n}$ and shown that there is a stable splitting

$$V_{n+m,n} \simeq \bigvee_{k=1}^{n} F_k^+ \tag{1}$$

where $F_k = F^k - F^{k-1}$ and $F_k^+ \cong F^k/F^{k-1}$ is the one-point compactification of F_k.

In particular, by taking $m = 0$ in (1), we have the following stable splitting of $U(n)$

$$U(n) \simeq \bigvee_{k=1}^{n} F_k(U(n))^+ \tag{2}$$

On the other hand it is known [3] that for a prime p there is a p-local unstable decomposition

$$U(n) \underset{p}{\simeq} X_1(n) \times \cdots \times X_{p-1}(n)$$

into a product of $p-1$ spaces, using the unstable Adams operations with the $X_i(n)$'s satisfying $H^*(X_i(n); \mathbb{Z}_{(p)}) \cong \Lambda_{\mathbb{Z}(p)}(x_i, x_{i+p-1}, \cdots, x_{i+s(p-1)})$ where $s = [\frac{n-i}{p-1}]$. Then we obtain a p-local stable splitting

$$U(n) \underset{p}{\simeq} \bigvee X_{i_1}(n) \wedge \cdots \wedge X_{i_s}(n),\ 1 \le i_1 < \cdots < i_s \le p-1. \tag{3}$$

2000 *Mathematics Subject Classification.* Primary 57T15; Secondary 55S99.
Key words and phrases. Stable splitting, Pontrjagin-Thom construction, Adams operation.
Received August 30, 2001.

In [4], Nishida and Yang have shown first, that the stable splitting map of unitary group $U(n)$ is homologically diagonal and then, by mixing (2) and (3), obtained a finer decomposition of $U(n)$.

In this paper, we will generalize the above argument to the case of the complex Stiefel manifold.

2. Property of stable splittings

We define an increasing filtration of the Stiefel manifold $V_{n+m,n}$

$$\{(1,0)\} = F^0 \subset F^1 \subset \cdots \subset F^n = V_{n+m,n}$$

by

$$F^k = \{(g,h) \in V_{n+m,n} |\ \dim_{\mathbb{C}}(\mathrm{Ker}(g-1))^{\perp} \le k\},\ 0 \le k \le n$$

where 1 denotes the unit element of $\mathrm{End}(\mathbb{C}^n)$. The difference $F^k - F^{k-1} = \{(g,h) \in V_{n+m,n} |\ \dim_{\mathbb{C}}(\mathrm{Ker}(g-1))^{\perp} = k\}$ is written by F_k. The one-point compactification F_k^+ is identified with the quotient space F^k/F^{k-1}. The pointed natural projection map is denoted by $\pi : (F^k)^+ \longrightarrow F^k/F^{k-1} = F_k^+$.

H.Miller [2] and M.Crabb [1] has shown the following

Theorem 2.1. *There exists a stable splitting*

$$V_{n+m,n} \simeq \bigvee_{k=1}^{n} F_k^+$$

Now we recall basic facts about the homology of F^k, see,e.g.,[5]. Cosider a map $\rho : \Sigma(\mathbb{C}P_+^{n+m-1}) \longrightarrow U(n+m)$ defined by

$$\rho(\lambda, z) = (\delta_{i,j} + (\lambda - 1) z_i \bar{z}_j), 1 \le i, j \le n+m$$

for $\lambda \in S^1, z = [z_1; \cdots ; z_{n+m}]$ and $z = (z_1, \cdots , z_{n+m}) \in S^{2n+2m-1} \subset \mathbb{C}^{n+m}$ and let $q : U(n+m) \longrightarrow U(n+m)/U(m) = V_{n+m,n}$ be the standard projection. Now let us consider the composition $\Sigma(\mathbb{C}P_+^{n+m-1}) \xrightarrow{\rho} U(n+m) \xrightarrow{q} U(n+m)/U(m) = V_{n+m,n}$. We have

$$H_*(V_{n+m,n}; \mathbb{Z}) \cong \mathbb{Z}\{x_{i_1} \cdots x_{i_s} | m+1 \le i_1 < \cdots < i_s \le m+n\}$$

where $x_i = (q \circ \rho)_*(\sigma(s_{i-1}))$ and s_{i-1} is a generator of $H_{2i-2}(\mathbb{C}P^{n+m-1}; \mathbb{Z})$. Let $j : F^k(U(n+m)) \longrightarrow U(n+m)$ and $j' : F^k \longrightarrow V_{n+m,n}$ be the inclusion maps. Consider the following commutative diagram

$$\begin{array}{ccc} F^k(U(n+m)) & \xrightarrow{\quad j \quad} & U(n+m) \\ \downarrow q & & \downarrow q \\ F^k & \xrightarrow{\quad j' \quad} & V_{n+m,n} \end{array}$$

Then from Prop 2.4 and Prop 2.5 [4], we have

Proposition 2.2. (i) *The homomorphism*

$$j'_* : H_*(F^k; \mathbb{Z}) \longrightarrow H_*(V_{n+m,n}; \mathbb{Z})$$

is injective and Im j'_* *is spanned by* $x_{i_1} \cdots x_{i_s}$, $m+1 \le i_1 < \cdots < i_s \le m+n$, $s \le k$.
(ii) *The homomorphism*

$$\pi_* : H_*(F^k; \mathbb{Z}) \longrightarrow \tilde{H}_*(F_k^+; \mathbb{Z})$$

is surjective and Ker π_* *is spanned by* $x_{i_1}\cdots x_{i_s}$, $m+1 \leq i_1 < \cdots < i_s \leq m+n$, $s \leq k-1$.

We write $\pi_*(x_{i_1}\cdots x_{i_k}) \in \tilde{H}_*(F_k^+;\mathbb{Z})$ by the same symbol $x_{i_1}\cdots x_{i_k}$. Now let $A \in F^k(U(n+m))$ and $B \in F^l$, then it is clear that the composition $AB \in F^{k+l}$. Thus we obtain a pairing

$$\tilde{\mu} : F^k(U(n+m)) \times F^l \longrightarrow F^{k+l}.$$

Note that $\tilde{\mu}(F^{k-1}(U(n+m)) \times F^l) \subset F^{k+l-1}$ and $\tilde{\mu}(F^k(U(n+m)) \times F^{l-1}) \subset F^{k+l-1}$. Therefore, by identifying $F_k(U(n+m))^+$ with $F^k(U(n+m))/F^{k-1}(U(n+m))$ and F_l^+ with F^l/F^{l-1}, we have an induced pairing

$$\tilde{\mu} : F_k(U(n+m))^+ \wedge F_l^+ \longrightarrow F_{k+l}^+.$$

Now we have

Proposition 2.3. *The diagram*

$$\begin{array}{ccc} F_k(U(n+m))^+ \wedge F_l^+ & \xrightarrow{\tilde{\mu}} & F_{k+l}^+ \\ \downarrow \sigma \wedge \varsigma & & \downarrow \varsigma \\ F^k(U(n+m))^+ \wedge (F^l)^+ & \xrightarrow{\tilde{\mu}} & (F^{k+l})^+ \end{array}$$

is homotopy commutative where σ *is the stable splitting map of* $U(n+m)$(*cf* [4]).

Proof. See the proof of the Proposition 2.6 [7].

Theorem 2.4. *The homomorphism*

$$\varsigma_* : \tilde{H}_*(F_l^+;\mathbb{Z}) \longrightarrow H_*(F^l;\mathbb{Z})$$

is given by

$$\varsigma_*(x_{j_1}\cdots x_{j_l}) = x_{j_1}\cdots x_{j_l}.$$

Proof. We prove the theorem by induction on l. It is clear for $l=1$. Suppose that it is true up to l. By Proposition 2.3 we have the following commutative diagram

$$\begin{array}{ccc} \tilde{H}_*(F_k^+;\mathbb{Z}) \otimes \tilde{H}_*(F_l^+;\mathbb{Z}) & \xrightarrow{\tilde{\mu}_*} & \tilde{H}_*(F_{k+l}^+;\mathbb{Z}) \\ \downarrow \sigma_* \otimes \varsigma_* & & \downarrow \varsigma_* \\ H_*(F^k;\mathbb{Z}) \otimes H_*(F^l;\mathbb{Z}) & \xrightarrow{\tilde{\mu}_*} & H_*(F^{k+l};\mathbb{Z}) \end{array}$$

It is clear that

$$\tilde{\mu}_*(x_{i_1}\cdots x_{i_k} \otimes x_{j_1}\cdots x_{j_l}) = \begin{cases} x_{i_1}\cdots x_{i_k} x_{j_1}\cdots x_{j_l} & ; m+1 \leq i_a, i_a \neq j_b \ \forall a,b \\ 0 & ; \text{otherwise} \end{cases}$$

We take $k=1$. Then $x_{j_1}\cdots x_{j_l} x_{j_{l+1}} = \tilde{\mu}_*(x_{j_1} \otimes x_{j_l}\cdots x_{j_{l+1}})$. Then $\varsigma_*(x_{j_1}\cdots x_{j_l} x_{j_{l+1}}) = x_{j_1}\cdots x_{j_l} x_{j_{l+1}}$ by the assumption of induction.

3. Adams operation and p-local stable splitttings

First of all, we recall the cohomology of $V_{n+m,n}$. Let $y_i \in H^{2i-1}(U(n+m);\mathbb{Z})$ be a class transgressive to $c_i \in H^{2i}(BU(n+m);\mathbb{Z})$. Then it is well known that y_i is primitive and we have

$$H^*(U(n+m);\mathbb{Z}) = \Lambda_{\mathbb{Z}}(y_1,\cdots,y_{n+m}).$$

Now consider the standard projection $p : U(n+m) \longrightarrow V_{n+m,n}$. Then the homomorphism $p^* : H^*(V_{n+m,n};\mathbb{Z})) \longrightarrow H^*(U(n+m);\mathbb{Z}))$ is injective and Im p^* is spanned by $y_{i_1}\cdots y_{i_s}$, $m{+}1{\le}i_1<\cdots<i_s{\le}m{+}n$. If we identify $H^*(V_{n+m,n};\mathbb{Z})$ with its image in $H^*(U(n{+}m);\mathbb{Z})$, then we may write $H^*(V_{n+m,n};\mathbb{Z}) = \Lambda_{\mathbb{Z}}(y_{m+1},\cdots,y_{n+m})$.

Let q be an integer such that $(q,m!)=1$. Then we know that there exists a map $\psi^q : BU(m) \longrightarrow BU(m)$ such that $(\psi^q)^*(c_r) = q^r c_r$, $1{\le}r{\le}m$. By applying the loop functor, we obtain a map

$$\Omega\psi^q : U(m) \longrightarrow U(m).$$

Now we consider the fibration $U(m) \to U(n+m) \to V_{n+m,n}$, then from the following diagram

$$\begin{array}{ccccccccc}
U(m) & \longrightarrow & U(n+m) & \longrightarrow & V_{n+m,n} & \longrightarrow & BU(m) & \longrightarrow & BU(n+m) \\
\downarrow{\scriptstyle \Omega\psi^q} & & \downarrow{\scriptstyle \Omega\psi^q} & & & & \downarrow{\scriptstyle \psi^q} & & \downarrow{\scriptstyle \psi^q} \\
U(m) & \longrightarrow & U(n+m) & \longrightarrow & V_{n+m,n} & \longrightarrow & BU(m) & \longrightarrow & BU(n+m)
\end{array}$$

we can show that there exists a map $V_{n+m,n} \to V_{n+m,n}$ such that all square diagrams are homotopy commutative. We denote this map by the same notation ψ^q and call it "Adams operation" too.

About the generators $y_{m+1},\cdots,y_{m+n}$, we have

Proposition 3.1. *The homomorphism*

$$(\psi^q)^* : H^*(V_{n+m,n};\mathbb{Z}) \longrightarrow H^*(V_{n+m,n};\mathbb{Z})$$

is given by

$$(\psi^q)^*(y_r) = q^r y_r$$

Proof. Since $(\Omega\psi^q) = q^r y_r$, the proposition follows from the above diagram.

Recall that the homology of $V_{n+m,n}$. Consider the standard projection $p : U(n+m) \longrightarrow V_{n+m,n}$. Let s_{i-1} be a generator of $H_{2i-2}(\mathbb{C}P^{n+m-1};\mathbb{Z})$ and let σ denote the homology suspension. Then we have

$$H_*(U(n+m);\mathbb{Z}) = \Lambda_{\mathbb{Z}}(x_1,\cdots,x_{n+m})$$

where $x_i = \rho_*(\sigma(s_{i-1})) \in$ is $H_{2i-1}(U(n+m);\mathbb{Z})$. (See [4].)

We write $p_*(x_{i_1}\cdots x_{i_s})$ by the same symbol $x_{i_1}\cdots x_{i_s}$. Now let p be an odd prime and $n+m$ be a positive integer. Then we can choose a prime l such that $(l,(n+m)!)=1$ and l generates the multiplicative group $\mathbb{Z}_p^\times$. In the ring $\{V_{n+m,n},V_{n+m,n}\}$

of homotopy classes of stable self maps, we can define, for each $t(1\leq t\leq p-1)$, a stable map $\phi_t : V_{n+m,n} \longrightarrow V_{n+m,n}$ by

$$\phi_t = \prod(\psi^l - l^i\text{id}),\ m+1\leq i\leq m+n \text{ and } i \not\equiv t \text{ mod } p-1$$

where the product is taken by means of composition.

Proposition 3.2. *$(\phi_t)_* : H_*(V_{n+m,n};\mathbb{Z}) \longrightarrow H_*(V_{n+m,n};\mathbb{Z})$ is given by*

$$(\phi_t)_*(x_{i_1}\cdots x_{i_s}) = \begin{cases} ax_{i_1}\cdots x_{i_s} & ;\, i_1+\cdots i_s \equiv t \textit{ mod } p-1 \\ 0 & ;\textit{ otherwise}\end{cases}$$

where a is a certain integer $\not\equiv 0$ mod p.

Proof. Consider $(\psi^l)^* : H^*(V_{n+m,n};\mathbb{Z}) \longrightarrow H^*(V_{n+m,n};\mathbb{Z})$. Since in general, in the ring $\{Z,Z\}$ of stable self maps of a spectrum Z, $(f+g)^*(z) = f^*(z) + g^*(z)$, we see $(\psi^l - l^i\text{id})^*(y_{i_1}\cdots y_{i_s}) = (l^{i_1+\cdots+i_s} - l^i)(y_{i_1}\cdots y_{i_s})$. Then clearly $(\phi_t)^*(y_{i_1}\cdots y_{i_s}) = \prod(l^{i_1+\cdots+i_s} - l^i)(y_{i_1}\cdots y_{i_s})$. Since $l^k - 1 \equiv 0$ mod p if and only if $k \equiv 0$ mod $p-1$, we see that $(\phi_t)^*$ satisfies the required property for the cohomology basis. Then the proposition follows from the duality.

Let $q_1, q_2, \cdots$ be all primes except p and put $d_k = q_1\cdots q_k$. Consider a sequence

$$V_{n+m,n} \xrightarrow{d_1} V_{n+m,n} \xrightarrow{\phi_t} V_{n+m,n} \xrightarrow{d_2} V_{n+m,n} \xrightarrow{\phi_t} \cdots$$

where d_k means the d_k-times of the identity. We denote by Y_t the telescope of the sequence. Note that the map $d_k : V_{n+m,n}\longrightarrow V_{n+m,n}$ is homologically diagonal. Let $\mu_t : V_{n+m,n}\longrightarrow Y_t$ be the natural inclusion. Then we have $(\mu_t)_*(x_{i_1}\cdots x_{i_s}) = 0$ for $i_1+\cdots+i_s \not\equiv t$ mod $p-1$ and writing $(\mu_t)_*(x_{i_1}\cdots x_{i_s})$ also by $x_{i_1}\cdots x_{i_s}$ for $i_1+\cdots+i_s \equiv t$ mod $p-1$. Then we have

Theorem 3.3. *The map*

$$\vee\mu_t : V_{n+m,n}\longrightarrow \bigvee_{t=1}^{p-1} Y_t$$

is a p-local equivalence.

Thus by mixing the above stable splitting and Miller's stable splitting, we have

Theorem 3.4. *Let n be a positive integer and let p be an odd prime. For each pair (t,k) of integers such as $1\leq t\leq p-1$ and $1\leq k\leq n$, $M_{t,k}$ denotes the submodule of $H_*(V_{n+m,n};\mathbb{Z}/p)$ spanned by $x_{i_1}\cdots x_{i_k}$ such that $i_1+\cdots+i_k \equiv t$ mod $p-1$. Then there exists a finite spectrum $Y_{t,k}$ satisfying*

$$H_*(Y_{t,k};\mathbb{Z}/p) \cong M_{t,k}$$

as a module, and a stable p-equivalence

$$V_{n+m,n} \longrightarrow \bigvee Y_{t,k}$$

where the wedge sum is taken over $1\leq t\leq p-1$ and $1\leq k\leq n$.

Proof. For $1\leq k\leq n$ and $1\leq t\leq p-1$, let e_k and f_t be the idempotents of the ring $\{U(n:m), U(n:m)\}$ coming from Miller's splitting and the one given by the Adams operation, respectively. Then we easily see that $(e_k)_*(f_t)_* = (f_t)_*(e_k)_*$ for all t,k. Let $Y_{t,k}$ be the telescope of the map $f_t\circ e_k$. Then Clearly $H_*(Y_{t,k};\mathbb{Z}) \cong \mathbb{Z}\{y_{i_1}\cdots y_{i_k} | i_1+\cdots i_k \equiv t \textit{ mod } p-1\}$. The latter half follows easily from what we have shown.

References

[1] M.Crabb, *On the stable splitting of* $U(n)$ *and* $\Omega U(n)$, Algebraic topology, Barcelona, pp35–53,(1986) Lecture Notes in Math., 1298, Springer, Berlin-New York, (1987).

[2] H.Miller, *Stable Splittings of Stiefel Manifolds*, Topology,Vol.24,No.4 pp.411-419 (1985)

[3] M.Mimura, G.Nishida and H.Toda, *Mod p Decomposition of Compact Lie Groups*,Publ. Res. Inst. Math. Sci. 13, no.3, pp627–680 (1977/78).

[4] G.Nishida and Y.Yang *On a p-local Stable Splitting of* $U(n)$, Journal of Mathematics of Kyoto University, to appear.

[5] N.E.Steenrod, *Cohomology Operation*, Princeton NewJersey, Princeton University Press (1962)

[6] D.Sullivan *Geometric topology, part I, Localization, periodicity and Galois symmetry*, Notes, M.I.T. (1970)

[7] Y.Yang *On a p-local Stable Splitting of Stiefel Manifolds*, Journal of the Mathematical Society of Japan , to appear.