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A BRIEF SURVEY OF THE GLUING FORMULA FOR THE ZETA-DETERMINANTS

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ABSTRACT. In this note we discuss two types of the gluing formula for the zeta-determinants of Laplacians which are known so far. For the first one, we explain the gluing formula with the Dirichlet and the Neumann boundary conditions. For the next one, we explain the adiabatic gluing formula with the Atiyah-Patodi-Singer boundary condition.

1. Introdcution

In case of global spectral invariants on a compact closed manifold, for example, eta-invariant and zeta-determinant of a Laplacian, people are interested in the gluing formula of those invariants as the same spirit with the Mayer-Vietoris sequence or the Seifert-Van Kampen theorem.

The gluing formula of an eta-invariant was shown by Bunke, Lesch, Bruning ([1], [2]) and by Wojciechowski in a sense of adiabatic limit ([7], [8]). In this note we briefly discuss the gluing formula for zeta-determinants which are known so far.

Let M be a compact Riemannian manifold with boundary Y (Y may be empty) and $E \to M$ be a complex vector bundle. We denote by A a positive definite, invertible differential operator. In case of $Y \neq \emptyset$, we choose an elliptic boundary condition B on Y, for example, Dirichlet (Neumann) boundary condition or Atiyah-Singer-Patodi boundary condition. We denote by A_B the same operator as A with the domain $Dom(A_B) = \{\phi \in C^\infty(E) | B(\phi) = 0\}$. Then A_B has discrete eigenvalues tending to ∞. For a simple argument, we assume that A_B is invertible.

Suppose that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \to \infty$ are the eigenvalues of A_B. Then

$$\zeta_{A_B}(s) = \sum_{i=1}^{\infty} \lambda_i^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} Tr e^{-tA_B} dt$$

is well defined for $Res > \frac{dim M}{ord(A)}$ and has a meromorphic continuation having a regular value at $s = 0$. We define the zeta-determinant $Det(A, B)$ of A with the boundary condition B by $Det(A, B) = e^{-\zeta'_{A_B}(0)}$.

In this note we explain the gluing formula given by Burghelea, Friedlander, Keppeler and the adiabatic gluing formula given by Jinsung Park and Wojciechowski.

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2. Gluing formula for Laplacians with the Dirichlet and the Neumann boundary conditions

We start from a simple example to see how the gluing formula looks like. Let (M, g) be a compact Riemannian manifold with non-empty boundary Y and $\tilde{M}$ be the smooth double of M, *i.e.*, $\tilde{M} = M \cup_Y M$. Let $\iota : \tilde{M} \to \tilde{M}$ be the natural involution sending a point in M to the same point in another copy. Then $\iota^2 = Id_{\tilde{M}}$ and the fixed point set of ι is Y. Then for a smooth section $\tilde{\phi}$ on $\tilde{M}$, we can write

$$\tilde{\phi}(x) = \frac{\phi(x) - \phi(\iota x)}{2} + \frac{\phi(x) + \phi(\iota x)}{2},$$

where $\phi = \tilde{\phi}|_M$. Then $\frac{\phi(x)-\phi(\iota x)}{2}$ satisfies the Dirichlet boundary condition and $\frac{\phi(x)+\phi(\iota x)}{2}$ satisfies the Neumann boundary condition on Y. Hence, $Spec(\Delta_{\tilde{M}}) = Spec(\Delta_{M,D}) \cup Spec(\Delta_{M,N})$, where $\Delta_{\tilde{M}}$ is the Laplacian on $\tilde{M}$ and $\Delta_{M,D}$, $\Delta_{M,N}$ are the Laplacians on M with the Dirichlet and the Neumann boundary conditions, respectively. Therefore, we obtain $\zeta_{\Delta_{\tilde{M}}}(s) = \zeta_{\Delta_{M,D}}(s) + \zeta_{\Delta_{M,N}}(s)$ and $logDet(\Delta_{\tilde{M}}) = logDet(\Delta_{M,D}) + logDet(\Delta_{M,N})$.

Now we are going to explain the gluing formula for Laplacians with the Dirichlet and the Neumann boundary conditions. Let M be a compact, closed Riemannian manifold and Y be a hypersurface of M so that $M - Y$ has two components. Denote by M_1, M_2 the closure of each component. Then $M = M_1 \cup_Y M_2$.

Suppose that A is a positive definite, invertible differential operator on M. For $z \in \{e^{i\theta} | -\pi < \theta < \pi\}$, we define a pseudodifferential operator $R(zt)$ ($t \in \mathbb{R}^+$) as the compositions of the following maps.

$$C^\infty(Y) \xrightarrow{\delta_{ia}} C^\infty(Y) \oplus C^\infty(Y) \xrightarrow{(P_1(zt), P_2(zt))} C^\infty(M_1) \oplus C^\infty(M_2)$$

$$\xrightarrow{(C_1, C_2)} C^\infty(Y) \oplus C^\infty(Y) \xrightarrow{\delta_{if}} C^\infty(Y).$$

Here $\delta_{ia}(\phi) = (\phi, \phi)$, $\delta_{if}(\phi, \psi) = \phi + \psi$ and $(C_1, C_2) = (\gamma_0 \partial_1, \gamma_0 \partial_1)$ with γ_0 the restriction map onto Y, ∂_1, ∂_2 inward normal derivative on the boundary of M_1, M_2, respectively. $P_1(zt)$, $P_2(zt)$ are the Poisson operators of $(A - zt)$ on M_1, M_2, respectively, which are characterized as follows.

$$\gamma_0 P_i(zt) = Id_Y, (A - zt) P_i(zt) = 0 on M_i, (i = 1, 2).$$

Now we consider $A^m + t^m = (A - \alpha_0 t)(A - \alpha_1 t) \cdots (A - \alpha_{m-1} t)$ with $dimM = m$, $\alpha_k = e^{i\frac{\pi + 2k\pi}{m}}$, $(0 \le k \le m-1)$ and $B_m = (\gamma_0, \gamma_0 A, \gamma_0 A^2, \cdots, \gamma_0 A^{m-1})$, $C_m = (C, CA, CA^2, \cdots, CA^{m-1})$ the the Dirichlet and the Neumann boundary conditions with respect to A^m or $A^m + t^m$.

Then the following theorem is due to Burghelea, Friedlander, Keppeler and the author ([3], [6]).

Theorem 2.1.
(1) $logDet(A^m + t^m) - logDet(A^m + t^m)_{M_1, B_m} - logDet(A^m + t^m)_{M_2, B_m} = -\sum_{k=0}^{m-1} c_k + \sum_{k=0}^{m-1} logDetR(\alpha_k t)$, *where* c_k *is the constant term in the asymptotic expansion of* $logDetR(\alpha_k t)$ *as* $t \to \infty$.

(2) $logDetA - logDetA_{M_1,D} - logDetA_{M_2,D} = -\frac{1}{m}\sum_{k=0}^{m-1} c_k + logDetR.$

Remark If the dimension of the manifold M is even, it is known that each constant c_k is zero ([5]).

Let M be a compact Riemannian manifold with boundary Y. For $z \in \{e^{i\theta} | -\pi < \theta < \pi\}$, define a pseudodifferential operator $Q(zt)$ $(t \in \mathbb{R}^+)$ by $Q(zt) = CP(zt) : C^\infty(Y) \to C^\infty(Y)$. By slightly modifying the above argument, one can prove the following fact.

Theorem 2.2.
(1) $logDet(A^m + t^m)_{M,C_m} - logDet(A^m + t^m)_{M,B_m} = -\sum_{k=0}^{m-1} d_k + \sum_{k=0}^{m-1} logDetQ(\alpha_k t)$, *where* d_k *is the constant term in the asymptotic expansion of* $logDetQ(\alpha_k t)$ *as* $t \to \infty$.

(2) $logDetA_{M,N} - logDetA_{M,D} = -\frac{1}{m}\sum_{k=0}^{m-1} d_k + logDetQ.$

Combining Theoren 2,1 and Theorem 2.2, we obtain the following corollary.

Corollary 2.3.
(1) $logDet(A^m + t^m) - logDet(A^m + t^m)_{M_1,B_m} - logDet(A^m + t^m)_{M_2,C_m} = -\sum_{k=0}^{m-1}(c_k - d_k) + \sum_{k=0}^{m-1} logDetR(\alpha_k t) - logDetQ(\alpha_k t)).$

(2) $logDetA - logDetA_{M_1,D} - logDetA_{M_2,N} = -\frac{1}{m}\sum_{k=0}^{m-1}(c_k - d_k) + logDetR - logDetQ.$

Remark (1) The operators $R(\alpha_k t)$ and $Q(\alpha_k t)$ depend not only on the boundary Y but also on M globally.

(2) In case of smooth double, ons can show that the right hand of each equation in Corollary 2.3 vanishes.

3. The gluing formula in the sense of adiabatic limit

Let (M, g) be a compact, closed Riemannian manifold and Y be a hypersurface of M so that $M - Y$ has two components. Denote by M_1, M_2 the closure of each component. Then $M = M_1 \cup_Y M_2$. Choose a collar neighborhood X of Y which is diffeomorphic to $Y \times (-1, 1)$. We assume that the metric g on M is a product metric on X. Suppose that D is a Dirac operator on M such that on X, $D^2 = -\partial_u^2 + B^2$, where ∂_u is a derivative along the normal direction to Y and B is a Dirac operator on Y. Then the spectrum of B is discrete and distributed from $-\infty$ to ∞. We denote by $\prod_> : C^\infty(Y) \to C^\infty(Y)$ the projection onto the positive eigensections of B and $\prod_<$ the projection onto the negative eigensections of B. These projections are called Atiyah-Patodi-Singer boundary condition.

Suppose that $M_R = (M - X) \cup (Y \times (-R, R))$, *i.e.*, M_R is obtained by attaching $Y \times (-R, R)$ to $M - X$. In this case, we extend the Dirac operator D on M to D_R on M_R in the natural way. The following theorem is due to Jinsung Park and K. Wojciechowski.

Theorem 3.1. *Assume that* $ker_{L^2} D_{1,\infty} = \{0\} = ker_{L^2} D_{2,\infty}$ *and* $kerB = \{0\}$. *Then*

$$\lim_{R\to\infty}(logDet(D_R^2) - logDet(D^2_{1,R,\prod_<}) - logDet(D^2_{2,R,\prod_>})) = -\zeta_{B^2}(0).$$

By using their argument, one can prove the following fact.

Theorem 3.2.

(1) Let M be a compact Riemannian manifold with boundary Y. Assume that the metric is product one near Y. Then

$$\zeta_{(\Delta_{M,N}+tId)}(0) - \zeta_{(\Delta_{M,D}+tId)}(0) = \frac{1}{2}\zeta_{(\Delta_Y+tId)}(0), t \in \mathbb{R}^+.$$

(2) Let M_1 and M_2 be compact Riemannian manifolds with common boundary Y. Assume that each metric is a product one near the boundary and $M = M_1 \cup_Y M_2$ is a smooth Riemannian manifold. Then

$$\lim_{t\to\infty} (logDet(\Delta_M + t) - logDet(\Delta_{M_1,D} + t) - logDet(\Delta_{M_2,N} + t)) = 0.$$

4. Some questions

We enclose this note by listing some questions.

(1) Can we compute the constant $-\sum_{k=0}^{m-1} c_k$ in Theorem 2.1 when $dimM$ is odd? Or can we determine whether this constant depends only on the boundary or not ?

(2) Can we make the formula in the Theorem 2.1 simpler under stronger assumptions of a manifold?

(3) Can we prove the similar ststements with the Dirichlet and the Neumann boundary conditions as the Theorem 3.1 by using the Theorem 2.1 ?

(4) Can we describe term $logDet(D_R^2) - logDet(D^2_{1,R,\prod_<}) - logDet(D^2_{2,R,\prod_>})) + \zeta_{B^2}(0)$ in the Theorem 3.1 in terms of R which vanishes as $R \to \infty$?

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