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SYMPLECTIC MANIFOLDS WITH HAMILTONIAN TORUS ACTIONS

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Abstract. In the first part of the paper, we introduce the definition of abstract moment maps on manifolds with torus actions and show the cobordism theory on these manifolds. In the second half, we consider symplectic manifolds with Hamiltonian tours actions. Let (M, ω) be a $2n$-dimensional symplectic manifold with a Hamiltonian S^1-action having isolated fixed points, and assume that the associated moment map μ is proper and bounded from below. Then M is cobordant to the disjoint union of vector spaces $\mathbb{C}^n$, over all fixed points in M, with appropriate symplectic forms and moment maps.

1. Abstract moment maps and cobordisms

Throughout this paper, M is a T-manifold (C^∞-smooth) with the compact Lie group T being a torus. As usual, $\mathfrak{t}$ denotes the Lie algebra of T, and X_ξ is the vector field induced by the action of $\xi \in \mathfrak{t}$ on M.

First of all, let us recall the definition of abstract moment maps, as given in [GGK1] and [K]. An *abstract moment map* on M is a T-equivariant smooth map $\phi : M \to \mathfrak{t}^*$ such that for any subgroup $H \subset T$, the composition of ϕ with the natural projection $\mathfrak{t}^* \to \mathfrak{h}^*$ is locally constant on the set M^H of points fixed by H. It is enough to assume that for any Lie algebra element $\xi \in \mathfrak{t}$, the map $\langle \phi, \xi \rangle : M \to \mathbb{R}$ is locally constant on the set of zeros of the corresponding vector field X_ξ.

For example, on a vector space on which the circle acts linearly and fixes only the origin, the map $\phi(v) = \|v\|^2$, with respect to any invariant metric, is an abstract moment map that is proper and bounded from below. Also, suppose that (M, ω) is a pre-symplectic manifold on which T acts and ϕ is a moment map. This means that ω is a closed T-invariant two-form on M and a T-equivariant map $\phi : M \to \mathfrak{t}^*$ satisfies $\iota(X_\xi)\omega = -d\langle \phi, \xi \rangle$ for all $\xi \in \mathfrak{t}$. Then ϕ is an abstract moment map. Moment maps on symplectic manifolds are among examples of abstract moment maps. In general, however, an abstract moment map is not associated with a symplectic form or even a closed two-form.

Abstract moment maps were introduced in [K] to define T-cobordisms and to state and prove the cobordism theorem. (See Theorem 1.1.)

We will work with the triple (M, ϕ, c) where M is a manifold on which a torus T acts, $\phi : M \to \mathfrak{t}^*$ is a proper abstract moment map and $c \in H_T^*(M; \mathbb{R})$ is an equivariant cohomology class on M. For example, (M, ω) is a symplectic manifold

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with a torus action, ϕ is a moment map and $c = [\omega + \phi]$ is an equivariant two cohomology class on M. In Section 2, we will investigate this triple (M, ω, ϕ).

Let M_1 and M_2 be smooth manifolds with actions of a torus T, let $\phi_1 : M_1 \to \mathfrak{t}^*$ and $\phi_2 : M_2 \to \mathfrak{t}^*$ be proper abstract moment maps, and let c_1 and c_2 be equivariant cohomology classes on M_1 and M_2, respectively. One says that (M_1, ϕ_1, c_1) and (M_2, ϕ_2, c_2) are *cobordant* if there exists a triple $(W, \tilde{\phi}, \tilde{c})$ where W is a T-manifold with boundary, $\tilde{\phi} : W \to \mathfrak{t}^*$ is a proper abstract moment map and $\tilde{c}$ is an equivariant cohomology class on W, such that ∂W is equal to the disjoint union $M_1 \sqcup M_2$ and such that the pullbacks of $\tilde{\phi}$ and $\tilde{c}$ to M_i are ϕ_i and c_i, $i = 1, 2$.

Theorem 1.1 [K]. *Let M be a manifold with an action of a torus T, let $\phi : M \to \mathfrak{t}^*$ be an abstract moment map, and let c be an equivariant cohomology class on M. Suppose that the vector $\eta \in \mathfrak{t}$ is such that the η-component of the abstract moment map, $\langle \phi, \eta \rangle : M \to \mathbb{R}$ is proper and bounded from below. Denote by M^η the zero set of the vector field on M corresponding to η. Then the triple (M, ϕ, c) is cobordant to the disjoint union, over all the components F of the set M^η, of the triples (NF, ϕ^η_F, c_F), where NF is the normal bundle of F in M with the T-action induced from that on M, c_F is the pullback of the cohomology class c via the maps $NF \to F \hookrightarrow M$, and ϕ^η_F is the T-abstract moment map*

$$\phi^\eta_F(v) = \phi(p) + \sum_i \|v_i\| \alpha^\eta_{i,F}, \quad v \in N_p F, \quad p \in F,$$

such that whose η-component $\langle \phi^\eta_F, \eta \rangle$ is proper and bounded from below. Here $\alpha^\eta_{i,F} \in \mathfrak{t}^*$ is the "polarized weight" with the sign chosen so that $\langle \alpha^\eta_{i,F}, \eta \rangle > 0$.

Consider the triple (M, ϕ, c) consisting of a manifold with a T-action, a proper abstract moment map and an equivariant cohomology class. We note that if a is a regular value of ϕ, then the T-action on the level set $\phi^{-1}(a)$ is locally free, that is, all stabilizers are discrete. Then the reduction $(M_{\mathrm{red}}, c_{\mathrm{red}})$ is produced where $M_{\mathrm{red}} := \phi^{-1}(a)/T$ is a compact orbifold and $c_{\mathrm{red}} \in H^*(M_{\mathrm{red}})$ such that $\pi^*(c_{\mathrm{red}}) = c|_{\phi^{-1}(a)} \in H^*_T(\phi^{-1}(a))$. Here $\pi : \phi^{-1}(a) \to M_{\mathrm{red}}$ is the quotient map.

Let M_1 and M_2 be compact orbifolds, and let c_1 and c_2 be cohomology classes on M_1 and M_2, respectively. (M_1, c_1) and (M_2, c_2) are said to be *cobordant* if there exist a compact orbifold W with boundary and a cohomology class $\tilde{c}$ on W, such that ∂W is the disjoint union $M_1 \sqcup M_2$ and such that the pullback of $\tilde{c}$ to M_i is c_i, $i = 1, 2$.

Guillemin was the first to note that cobordism commutes with reduction. That is,

Theorem 1.2. *Let (M, ϕ, c) and (M', ϕ', c') be cobordant manifolds with T-actions, proper abstract moment maps and equivariant cohomology classes. Let $a \in \mathfrak{t}^*$ be a regular value of ϕ and ϕ'. Then the corresponding reductions $(M_{\mathrm{red}}, c_{\mathrm{red}})$ and $(M'_{\mathrm{red}}, c'_{\mathrm{red}})$ are cobordant.*

2. Cobordisms of symplectic manifolds with Hamiltonian torus actions

Now let us apply the discussions in Section 1 to symplectic manifolds with T-actions and with moment maps. Let (M_1, ω_1) and (M_2, ω_2) be $2n$-dimensional symplectic manifolds on which a torus T acts with associated moment maps $\mu_1 : M_1 \to \mathfrak{t}^*$ and $\mu_2 : M_2 \to \mathfrak{t}^*$. (M_1, ω_1, μ_1) and (M_2, ω_2, μ_2) are said to be *cobordant* if there exists a $(2n+1)$-dimensional manifold W with boundary and a close two-form $\tilde{\omega}$ with a moment map $\tilde{\mu} : W \to \mathfrak{t}^*$ such that $\partial W = M_1 \sqcup M_2$, and such that the pullbacks of $\tilde{\omega}$ and $\tilde{\mu}$ to M_i are ω_i and μ_i, $i = 1, 2$.

Let (M, ω) be a symplectic manifold with a T-action and a proper moment map μ. If a is a regular value of μ, then the T-action on $\mu^{-1}(a)$ is locally free. We denote by M_{red} the quotient space $\mu^{-1}(a)/T$. There is a closed two-form ω_{red} on M_{red} which is characterized by the equality $\pi^*(\omega_{\text{red}}) = \omega|_{\mu^{-1}(a)}$ where $\pi : \mu^{-1}(a) \to M_{\text{red}}$ is the quotient map. Hence, M_{red} is a compact symplectic orbifold. In particular, if the T-action on the regular level set $\mu^{-1}(a)$ is free, then $(M_{\text{red}}, \omega_{\text{red}})$ is a compact symplectic manifold. We note that the Kirwan map $H_T^*(M) \to H^*(M_{\text{red}})$ maps $[\omega + \mu]$ to $[\omega_{\text{red}} + a]$.

Theorem 2.1. *Cobordism commutes with reduction : Let $(W, \tilde{\omega}, \tilde{\mu})$ be a cobordism between (M, ω, μ) and (M', ω', μ'), and let $a \in \mathfrak{t}^*$ be a regular value of $\tilde{\mu}$. Then $(W_{\text{red}}, \tilde{\omega}_{\text{red}})$ is a cobordism between $(M_{\text{red}}, \omega_{\text{red}})$ and $(M'_{\text{red}}, \omega'_{\text{red}})$.*

Let $(\mathbb{C}^n, \omega_0)$ be a symplectic manifold with the standard symplectic form

$$\omega_0 = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j.$$

Consider the S^1-action on $\mathbb{C}^n$ by multiplication

$$t \cdot (z_1, \dots, z_n) = (t\, z_1, \dots, t\, z_n).$$

Then it is Hamiltonian with moment map

$$\mu(z) = \frac{1}{2} \sum_{j=1}^{n} |z_j|^2,$$

and all non-zero real numbers are regular values of μ.

From now on, consider that (M, ω) is a $2n$-dimensional symplectic manifold with a Hamiltonian S^1-action and with the associated moment map $\mu : M \to \mathbb{R}$. Assume that S^1 acts with isolated fixed points $\{p_1, p_2, \dots, p_k, \dots\}$ and that μ is proper and bounded from below. Theorem 1.1 implies that M is cobordant to the disjoint union of the vector space $T_{p_k}M$ over all fixed points p_k in M. Also, there is an isomorphism $T_{p_k}M \simeq \mathbb{C}^n$. Hence we consider the S^1-action on $\mathbb{C}^n$:

$$e^{i\theta} \cdot (z_1, \dots, z_n) = (e^{i\theta\alpha_1^k} z_1, \dots, e^{i\theta\alpha_n^k} z_n), \quad \alpha_j^k \neq 0 \in \mathbb{Z}.$$

Therefore, we obtain the following theorem:

THEOREM 2.2. *Let (M, ω, μ) be the same as above. Then M is cobordant to the disjoint union of vector spaces $\mathbb{C}^n$ with the symplectic form*

$$\omega_k = \frac{i}{2} \sum_{j=1}^{n} sign(\alpha_j^k) dz_j \wedge d\bar{z}_j$$

and the associated moment map

$$\mu_k(z) = \frac{1}{2} \sum_{j=1}^{n} |\alpha_j^k||z_j|^2 + \mu(p_k)$$

which is obviously bounded from below and proper.

By Theorem 2.1 and Theorem 2.2, we also obtain the following theorem:

THEOREM 2.3. *Let a is a regular value of μ and all the μ_k's. Set $M_{\rm red} := \mu^{-1}(a)/T$ and $M_{k,\rm red} := \mu_k^{-1}(a)/T$, $k = 1, 2, \ldots$. Then $M_{\rm red}$ is cobordant to the disjoint union of compact symplectic orbifolds $M_{k,\rm red}$.*

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