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SURVEY ON CONTACT 3-MANIFOLDS

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Abstract. We try to survey the recent results about contact structures on 3-manifolds.

1. Introduction

A contact structure ξ on a 3-manifold M is a 2-plane field which is completely non-integrable. The complete non-integrability of ξ can be expressed by the inequality $\alpha \wedge d\alpha \neq 0$ (nowhere vanishing) for a 1-form α which defines (at least locally) the plane field ξ, i.e., $\xi = \{\alpha = 0\}$. Note that the sign of $\alpha \wedge d\alpha$ gives an orientation of M. If M is already oriented, we can compare these two orientations, and can distinguish between *positive* and *negative* contact structures.

Contact structures fall into one of two classes; tight or over-twisted. To define tight and over-twisted contact structures, we need to investigate embedded surfaces in a contact manifold. Let $F^2 \subset M^3$ be a 2-surface in a contact 3-manifold (M, ξ). Generically F is tangent to ξ at a finite set $\Sigma = \{p_1 \dots p_k\} \subset F$. Outside Σ the contact structure ξ intersects the tangent field along a line field K, which integrates to a 1-dimensional foliation on F with singularities at points of Σ. This singular foliation on F is called *characteristic foliation* of F, and denoted by F_ξ.

Definition 1.1. *A contact structure ξ on M is called* over-twisted *if there exists an embedded 2-disk $D \subset M$ such that D_ξ contains a limit cycle. (See figure 1) If a contact structure ξ is not over-twisted , it is called* tight

Figure 1. Over-twisted disk

2. Early Works

In 1971, Martinet [M] showed that any orientable closed 3-manifold admits a contact structure. Three years later after the work of R. Lutz [L] and in the wake of triumph of Gromov's h-principle, it seemed that the classification of closed contact 3-manifolds was at hand. Ten years later in the seminal work [B], D. Bennequin

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showed that the situation is much more complicated and that the classification of contact structures on 3-manifolds, and even on S^3, was not likely to be achieved. [1]

After Martinet's work of 1971, through the work of Bennequin [B] and Eliashberg [E2], it became apparent that not all contact structures are created equal in dimension 3. Specially, they found that contact structures fall into two classes, which are tight and over-twisted. In this new light, what Martinet actually showed was that every 3-manifold admits an over-twisted contact structure.

3. Work of early 1990's

In 1989, Eliashberg made some progress in [E1]. In this paper, Eliashberg classified over-twisted contact structures on closed 3-manifolds by showing the weak homotopy equivalence of the space of over-twisted contact structures up to isotopy and the space of 2-plane fields up to homotopy - hence overtwisted contact structures could be understood via homotopy theory.

On the other hand, understanding tight contact structures is not as simple as that of over-twisted contact structures. It has become apparent that tight contact structures have deep connection to the topology of 3-manifolds, not limited to homotopy. For example, Eliashberg and Thurston found connections to foliation theory in [EnT]. Rudoph [R] and Lisca and Matić [LM] found connections with slice knots and slice genus. Kronheimer and Mrowka [KM] found connections with Seiberg-Witten theory. Thus understanding tight contact structures became a central issue in 3-dimensional contact topology.

In 1992, Eliashberg classified contact structures on S^3 in [E2]. This is the first example of the classification of tight contact structures on 3-manifold.

Theorem 3.1 ([E2], Classification of tight contact structures on S^3)**.** *A tight contact structure on S^3 is isotopic to the standard contact structure, hence it is unique up to isotopy.*

Homotopy classes of 2-plane fields on S^3 can be canonically identified with homotopy classes from $\pi_3(S^2) = Z$ and, therefore, can be numbered by integers: $\alpha_0, \alpha_{\pm 1}, \dots$. With these notations, we have

Theorem 3.2 ([E2], Classification of contact structures on S^3)**.** *The class α_0 contains exactly two non-equivalent (positive) contact structures : the standard and the over-twisted. All other classes $\alpha_i, |i| > 0$, contain only one contact structure, the over-twisted.*

4. Recent Works

For couple of years after Eliashberg's work, S^3 had been the only known example, and in 1996 J. Etnyre made a breakthrough. In [ET1], Etnyre introduced a new technique to classify tight contact structures on some 3-manifolds, and by using this technique, he found the classification of $L(3,1)$ and $L(3,2)$. And this paper became a corner stone for the later results relating to the the classification of tight contact structures. The results are followings.

Theorem 4.1 ([ET1])**.** *$L(3,1)$ has exactly two tight contact structures, one for each non-zero element of $H^2(L(3,1), Z)$, and $L(3,2)$ has a unique tight contact structure, that is for zero element in $H^2(L(3,2), Z)$*

[1] quoted from [E2]

Etnyre's new method worked on $L(3,q)$, but it was not easily applied to other 3-manifolds, even to lens spaces. In 1999, Sungjoon Ko gave classifications of tight contact structures on some lens spaces by using Etnyre's method and geometric intuition [Ko].

Theorem 4.2 ([Ko]). *$L(p,p-1)$, p is odd, has a unique tight contact structure, and this realizes the zero element in $H^2(L(p,p-1),Z)$.*

At the same time, Honda Ko [HK1] and Giroux [G2] classified tight contact structures on various 3-manifolds independently. Here, I briefly introduce the results of Honda Ko in [HK1], [HK2] and [HK3].

Honda's method is a systematic application of the methods developed by Kanda [K], which in turn use Giroux's theory of convex surfaces [G1]. In essence, he use Kanda's methods and apply them in Etnyre's setting; decompose the 3-manifold M in a series of steps, along closed convex surfaces or convex surfaces with *Legendrian* boundary. The difference between Etnyre's approach and Honda's one is that Honda's require that the cutting surfaces have boundary consisting of *Legendrian* curves, whereas Etnyre used cutting surfaces which had transverse curves on the boundary. Fortunately, *Legendrian* curve approach appears to be more efficient than the transverse curve approach. The followings are Honda's results using *Legendrian* boundary.

Theorem 4.3 ([HK1] Classification on lens spaces). *Let $L(p,q)$ be a lens space, where $p > q > 0$ and $(p,q) = 1$. Assume $-\frac{p}{q}$ has the continued fraction expansion*

$$-\frac{p}{q} = r_0 - \frac{1}{r_1 - \frac{1}{r_2 - \cdots \frac{1}{r_k}}}$$

with all $r_k < -1$. Then there exist exactly $|(r_0+1)(r_1+1)\cdots(r_k = 1)|$ tight contact structures on $L(p,q)$ up to isotopy. Moreover, all the tight contact structures on $L(p,q)$ can be obtained from Legendrian surgery on links in S^3, and are therefore holomorphically fillable.

He gave the similar results on $T^2 \times I$ and on Solid tori in the same paper [HK1]. In [HK2], Honda also classified tight contact structures on T^2 bundles over S^1. Since the results can not be expressed in a short formula, we skip the detail results. See [HK2] if interested. And the third paper of the trilogy [HK3] devoted to circle bundles over closed Riemann surfaces.

Giroux has independently obtained similar classification results in [G2]. Interesting fact is that Honda's approach and Giroux's approach are surprisingly dissimilar. So it is worth reading [G2] also.

Another interesting topic of the tight contact structure is the existence. In [EH1], Etnyre and Honda exhibited a 3-manifold which admits no tight contact structure.

Theorem 4.4 ([EH1]). *There exist no positive tight contact structures on the Poincaré homology sphere $\Sigma(2,3,5)$ with reverse orientation.*

This was the first example of a closed 3-manifold which does not carry a positive tight contact structure.

Corollary 4.5 ([EH1]). *Let M be the Poincaré homology sphere with reverse orientation. Then the connect sum $M\#\bar{M}$, where $\bar{M}$ is M with the opposite orientation, does not carry any tight contact structure, positive or negative.*

Usually, showing tightness is not easy, since we have to show that it does not have any embedded over-twisted disk. Gromov's theory of holomorphic curves [Gr] make it easier to determine if a contact structure on a 3-manifold is tight. Roughly speaking, a contact structure is *symplectically fillable* if it is the boundary of a symplectic 4-manifold. Gromov and Eliashberg showed that a symplectically fillable structure was necessarily tight. In fact, until mid-1990's, all known tight contact structures were shown to be tight using symplectic fillings. This included two rich sources of tight contact structures - perturbations of taut foliations as in [EnT], and *Legendrian* surgery as in [E3] and [W]. This promoted Eliashberg and others to ask whether tight contact structures are the same as symplectically fillable contact structures. Subsequently, gluing techniques were developed by Colin and Makar-Limanov [ML], and strengthened in [HK4]. Largely due to the improvements in gluing techniques, tight contact structures could be constructed without resorting to symplectic filling techniques. In [EH2], Etnyre and Honda constructed a manifold with a tight contact structure, which is not symplectically fillable.

As we mentioned, the easiest way to prove a contact structure is tight is to show it 'bounds' a symplectic 4-manifold. There are several notions of 'symplectic filling', and let me assemble the various notions here. (See [EH2] for the detail). A contact structure ξ on M is :

1. *Holomorphically fillable* if (M, ξ) is the ω-convex boundary of some Stein manifold (X, ω).
2. *Strongly symplectically fillable* if (M, ξ) is the ω-convex boundary of some symplectic manifold (X, ω).
3. *Weakly symplectically fillable* if (M, ξ) is the weakly convex boundary of some symplectic manifold (X, ω).
4. *Weakly symplectically semi-fillable* if (M, ξ) is one component of the weakly convex boundary of some symplectic manifold (X, ω).

Theorem 4.6 (Gromov-Eliashberg)**.** *Let (M, ξ) be a contact 3-manifold which satisfies any of the above conditions for fillabiliy, Then ξ is tight.*

The following diagram indicates the hierarchy of contact structures.

<table>
<tr><td>Tight</td><td></td><td></td></tr>
<tr><td>$\cup\!\!\!/$</td><td></td><td></td></tr>
<tr><td>Weakly symplectically semi-fillable</td><td>$\supsetneq$</td><td>Strongly symplectically semi-fillable</td></tr>
<tr><td>$\cup$</td><td></td><td>$\cup$</td></tr>
<tr><td>Weakly symplectic fillable</td><td>$\supsetneq$</td><td>Strongly symplectically fillable</td></tr>
<tr><td></td><td></td><td>$\cup$</td></tr>
<tr><td></td><td></td><td>Holomorphically fillable</td></tr>
</table>

The proper inclusion of the set of weakly symplectically semi-fillable contact structures into the set of tight contact structures is showed in [EH2]. The proper inclusion of the set of strongly fillable structures into the set of weakly fillable structures is done by Eliashberg in [E4]. For all other inclusions in the diagram it is not known whether the inclusions are strict. [2]

[2]quoted from [EH2]

References

[B] D. Bennequin, *Entrelacements et equations de Pfaff*, Astérique **107 108**, (1983), 83-161

[E1] Y. Eliashberg, *Classification of over-twisted contact structures on 3-manifolds*, Invent. Math. **98** (1989), 623–637.

[E2] Y. Eliashberg, *Contact 3-manifolds twenty years since J. Martinet's work* , Ann. Inst. Fourier **42** (1992), 165–192.

[E3] Y. Eliashberg, *Topological characterization of Stein manifolds of dimension ¿ 2*, Int. J. of Math **1** (1990), 29-46.

[E4] Y. Eliashberg, *Unique holomorphically fillable contact structure on the 3-torus*, Internat. Math. Res. Notice **2** (1996), 77-82.

[EnT] Y.Eliashberg and W. Thurston, *Confoliations*, AMS Univ. Lec. Series **Vol 13** (1998).

[ET1] J. Etnyre, *Symplectic Constructions on 4-manifolds*, Dissertation, University of Texas, (1996).

[ET2] J. Etnyre, *Tight Contact Structures on Lens Spaces*, Preprint, (1997).

[EH1] J. Etnyre and K. Honda, *On the non-existence of tight contact structures*, Preprint,(1999)

[EH2] J. Etnyre and K. Honda, *Tight contact structures with no symplectic fillings*, Preprint,(2000)

[G1] E. Giroux, *Convexité en topologie de contact*, Comment. Math. Helvetici **66** (1991), 637-677

[G2] E. Giroux, *Structure de contact en dimension trois and bifurcations des feuilletages de surfaces*, preprint (1999)

[Gr] M. Gromov, *Pseudo-holomorphic curves in symplectic manifolds*, Invent. Math **82** (1985), 307-347

[HK1] K. Honda, *On the classification of tight contact structures I*, preprint, (2000).

[HK2] K. Honda, *On the classification of tight contact structures II*, to appear in J. Diff. Geom.

[HK3] K. Honda, *On the classification of tight contact structures III*, preprint, (2000).

[HK4] K. Honda, *Gluing tight contact structures*, preprint, (2000).

[K] Y. Kanda, *The classification of tight contact structures on the 3-torus*, Comm. in Anal. and Geom. **5** (1997), 413-438.

[Ko] S. Ko, *More about Tight Contact Structures on Lens Space* , Dissertation, University of California at Berkeley, (2000).

[KM] P. Kronheimer and T. Mrowka, *Monopoles and contact structures*, Invent. Math. **130** (1997), no 2, 209-255.

[L] R. Lutz, *Structures de contact sur les fibre's principaux en cercles de dimension 3* , Ann. Inst. Fourier, **27-3** (1977), 1-15.

[LM] P. Lisca and G. Matić, *Stein 4-manifolds with boundary and contact structures*, Top. and its App. **88** (1998), 55-66.

[M] J. Martinet, *Formes de contact sur les variétés de dimension 3*, Springer Lecture notes in Math **209**, (1971) 142-163

[ML] S. Makar-Limanov, *Morse surgeries of index 0 on tight manifolds*, preprint, (1997)

[R] L. Rudolph, *An obstruction to sliceness via contact geometry and classical gauge theory*, Invent. math. **119** (1995), 155-163.

[W] A. Weinstein, *Contact surgery and symplectic handlebodies*, Hokkiado. Math. Journal **20** (1991), 241-251.

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