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AN ALMOST KÄHLER STRUCTURE ON THE DEFORMATION SPACE OF CONVEX REAL PROJECTIVE STRUCTURES

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Abstract. In this article we summarize some results about the deformation space of (G, X)-structures on a smooth manifold and compare the hyperbolic structures and convex real projective structures. We also provide some partial results toward *Goldman conjecture*; The deformation space of convex real projective structures on a compact oriented surface with negative Euler characteristic has a Kähler structure.

1. Introduction

An real projective structure on a smooth n-manifold M is a system of coordinate charts in $\mathbb{RP}^n$ with transition maps in $\mathbf{PGL}(n+1, \mathbb{R})$. If $\chi(M) < 0$, then an equivalence classes of real projective structure on M form a deformation space $\mathbb{RP}^n(M)$. The study of $\mathbb{RP}^2$ structures has been quite active. Ehresmann, Kuiper, Benzécri, Kobayashi, and Thurston have done important work. Recently Goldman and Choi lead this field. Goldman [4] showed the component of $\mathbb{RP}^2(M)$ containing the *Teichmüller space* is a real manifold of dimension $-8 \cdot \chi(M)$. Above component is called the *Goldman space* $\mathcal{G}(M)$ and it is the deformation space of *convex* real projective structures. Goldman [5] also proved $\mathcal{G}(M)$ is a symplectic manifold. In this paper we survey some theorems about (G, X)-structure on M and provide Darvishzadeh-Goldman [2] and Loftin [8]'s partial results toward Goldman conjecture; $\mathcal{G}(M)$ is a Kähler manifold.

2. (G, X)-structures on a smooth manifold M

In this paper we assume that the action of a group G on a topological space X is *strongly effective*; that is, if $g_1, g_2 \in G$ agree on a nonempty open set of X, then $g_1 = g_2$. By this requirement, for any nontrivial $g \in G$, the set of fixed points $X_g = \{x \in X \mid gx = x\}$ is nowhere dense in X.

Example 2.1. Let $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ the upper half complex plane. Then $\mathbf{SL}(2, \mathbb{R})$ acts on $\mathbb{H}^2$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d} .$$

Since $Az = -Az$ for any $A \in \mathbf{SL}(2, \mathbb{R})$ and $z \in \mathbb{H}^2$, $\mathbf{PSL}(2, \mathbb{R}) = \mathbf{SL}(2, \mathbb{R})/\pm I$ acts strongly effectively on $\mathbb{H}^2$.

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Example 2.2. Let $\mathbb{RP}^2$ be the space of all lines through the origin in $\mathbb{R}^3$; if v is a nonzero vector in $\mathbb{R}^3$, then the corresponding point in $\mathbb{RP}^2$ will be denoted by $[v]$. Then $\mathbf{GL}(3,\mathbb{R})$ acts on $\mathbb{RP}^2$ by $A \cdot [v] = [Av]$. Since the scalar matrices $\mathbb{R}^* \subset \mathbf{GL}(3,\mathbb{R})$ acts trivially on $\mathbb{RP}^2$, $\mathbf{PGL}(3,\mathbb{R}) = \mathbf{GL}(3,\mathbb{R})/\mathbb{R}^*$ acts strongly effectively on $\mathbb{RP}^2$.

Let Ω be an open subset of X. A map $\phi : \Omega \to X$ is called *locally*-(G, X) if for each component $W \subset \Omega$, there exists $g \in G$ such that $\phi|_W = g|_W$.

Definition 2.3. Let X be a smooth n-manifold and G a connected algebraic Lie group acting strongly effectively on X. A (G, X)-*structure* on M is a maximal collection of $\{(U_\alpha, \psi_\alpha)\}$ such that

1. $\{U_\alpha\}$ is an open covering of M.
2. For each α, $\psi_\alpha : U_\alpha \to X$ is a diffeomorphism onto its image.
3. If (U_α, ψ_α) and (U_β, ψ_β) are two coordinate charts with $U_\alpha \cap U_\beta \neq \emptyset$, then $\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(U_\alpha \cap U_\beta) \to \psi_\beta(U_\alpha \cap U_\beta)$ is locally-(G, X).

A smooth n-manifold with a (G, X)-structure is called a (G, X)-manifold.

Let M be a compact oriented Riemann surface with $\chi(M) < 0$. $\big(\mathbf{PSL}(2,\mathbb{R}), \mathbb{H}^2\big)$-structures on M are called *hyperbolic structures* and $\big(\mathbf{PGL}(3,\mathbb{R}), \mathbb{RP}^2\big)$-structures on M are called *real projective structures*.

For a smooth n-manifold M, consider $\mathcal{A}(M) = \{(f, N)\}$ where $f : M \to N$ is a diffeomorphism and N is a (G, X)-manifold. We say two pairs (f, N) and (f', N') are *equivalent* if there exists diffeomorphism $k : N \to N'$ such that f' is isotopic to $k \circ f$. The set of equivalence classes of (G, X)-structures $\mathcal{D}(M) = \mathcal{A}(M)/\sim$ is called the *deformation space* of (G, X)-structures on M.

Definition 2.4. The deformation space of hyperbolic structures on M is called the *Teichmüller space* and denoted by $\mathfrak{T}(M)$. The deformation space of real projective structures on M is denoted by $\mathbb{RP}^2(M)$.

Let M and N be (G, X)-manifolds and $f : M \to N$ a smooth map. Then f is called a (G, X)-*map* if for each coordinate chart (U, ψ_U) on M and (V, ψ_V) on N, the composition $\psi_V \circ f \circ \psi_U^{-1} : \psi_U(f^{-1}(V) \cap U) \to \psi_V(f(U) \cap V)$ is locally-(G, X). The following famous theorem is due to Ehresmann [3] and Thurston [10].

Theorem 2.5. *Let M be an (G, X)-manifold and $p : \tilde{M} \to M$ denotes a fixed universal covering of M. Let π be the corresponding group of covering transformations.*

1. *There exist a (G, X)-map* $\mathbf{dev} : \tilde{M} \to X$ *and a homomorphism $h : \pi \to G$ such that for each $\gamma \in \pi$ the following diagram commutes.*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\mathbf{dev}} & X \\ {\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle h(\gamma)} \\ \tilde{M} & \xrightarrow[\mathbf{dev}]{} & X \end{array}$$

2. *Suppose* $(\mathbf{dev}', h')$ *is another pair satisfying above condition, then there exists $g \in G$ such that* $\mathbf{dev}' = g \circ \mathbf{dev}$ *and $h' = \iota_g \circ h$ where $\iota_g : G \to G$ denotes the*

inner automorphism defined by g; i.e. $h'(\gamma) = (\iota_g \circ h)(\gamma) = g \circ h(\gamma) \circ g^{-1}$.

$$\begin{array}{ccccc} \tilde{M} & \xrightarrow{\mathbf{dev}} & X & \xrightarrow{g} & X \\ \gamma\downarrow & & \downarrow h(\gamma) & & \downarrow h'(\gamma) \\ \tilde{M} & \xrightarrow[\mathbf{dev}]{} & X & \xrightarrow[g]{} & X \end{array}$$

The (G, X)-map $\mathbf{dev} : \tilde{M} \to X$ is called a *developing map*, the homomorphism $h : \pi \to G$ is called the *holonomy homomorphism*, the image $\Omega = \mathbf{dev}(\tilde{M}) \subset X$ is called the *developing image,* and the image $\Gamma = h(\pi) \subset G$ is called the *holonomy group.*

3. Comparison between hyperbolic and real projective structures

In hyperbolic structures case the developing map $\mathbf{dev} : \tilde{M} \to \Omega$ is a diffeomorphism and the holonomy homomorphism $h : \pi \to \Gamma$ is an isomorphism such that Γ is a discrete group acting properly and freely on Ω. But in real projective structures case it is not; we can find examples in Sullivan and Thurston's paper [9].

A domain $\Omega \subset \mathbb{RP}^2$ is called *convex* if there exist a projective line $l \subset \mathbb{RP}^2$ such that $\Omega \subset (\mathbb{RP}^2 - l)$ and Ω is a convex subset of the affine plane $\mathbb{RP}^2 - l$; i.e. if $x, y \in \Omega$ then the line segment $\bar{xy}$ lies in Ω. By definition, $\mathbb{RP}^2$ itself is not convex.

Definition 3.1. A real projective structure on M is called *convex* if $\mathbf{dev}$ is a diffeomorphism onto a convex domain in $\mathbb{RP}^2$.

Proposition 3.2 (Goldman [4])**.** *Let M be an $\mathbb{RP}^2$-manifold. Then the following statements are equivalent.*

1. *M has a convex $\mathbb{RP}^2$-structure.*
2. *M is diffeomorphic to a quotient Ω/Γ where $\Omega \subset \mathbb{RP}^2$ is a convex domain and $\Gamma \subset \mathbf{PGL}(3, \mathbb{R})$ is a discrete group acting properly and freely on Ω.*

If M is a convex $\mathbb{RP}^2$-manifold, then we can identify $M = \Omega/\Gamma$ where Ω is the developing image and Γ the holonomy group.

Definition 3.3. The *Goldman space* $\mathcal{G}(M)$ is the subset of $\mathbb{RP}^2(M)$ corresponding to the deformation space of *convex* $\mathbb{RP}^2$-structures.

Let $A \in \mathbf{SL}(2, \mathbb{R})$, then A is said to be *hyperbolic* if it has two distinct real eigenvalues. Since $f(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + 1$ is the characteristic polynomial of A, it is equivalent to say that $\mathrm{tr}(A)^2 - 4 > 0$.

Let $A \in \mathbf{PSL}(2, \mathbb{R})$, then the absolute value of $\mathrm{tr}(A)$ is defined. A is said to be *hyperbolic* if $|\mathrm{tr}(A)| > 2$. A can be expressed by the diagonal matrix

$$\pm \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

via an $\mathbf{SL}(2, \mathbb{R})$-conjugation where $0 < \alpha < 1$.

The homomorphism $\mathbf{GL}(3, \mathbb{R}) \to \mathbf{SL}(3, \mathbb{R})$ defined by $A \mapsto (\det A)^{-1/3} A$ induces an isomorphism $\mathbf{PGL}(3, \mathbb{R}) \to \mathbf{SL}(3, \mathbb{R})$ as *analytic* groups. Thus from now on we shall identify the groups $\mathbf{PGL}(3, \mathbb{R})$ and $\mathbf{SL}(3, \mathbb{R})$.

Let $A \in \mathbf{SL}(3,\mathbb{R})$, then A is said to be *positive hyperbolic* if it has three distinct positive real eigenvalues. A can be expressed by the diagonal matrix

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix} \tag{1}$$

via an $\mathbf{SL}(3,\mathbb{R})$-conjugation where $\lambda\mu\nu = 1$ and $0 < \lambda < \mu < \nu$.

Proposition 3.4. *Let M be a hyperbolic manifold. Then each nontrivial element of holonomy group $\Gamma \subset \mathbf{PSL}(2,\mathbb{R})$ is hyperbolic.*

Proposition 3.5 (Kuiper [7])**.** *Let M be a convex real projective manifold. Then each nontrivial element of holonomy group $\Gamma \subset \mathbf{SL}(3,\mathbb{R})$ is positive hyperbolic.*

Then we can derive that the real projective structures are an extension of the hyperbolic structures through the identification

$$\pm \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \leftrightarrow \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha^{-2} \end{pmatrix}. \tag{2}$$

Let $A \in \mathbf{PSL}(2,\mathbb{R})$ be a hyperbolic element. We define length parameter ℓ of A by

$$\ell(A) = \log(\alpha^{-4}) = 4\cosh^{-1}(\frac{|\mathrm{tr}(A)|}{2}).$$

Then ℓ is a coordinate in the set of hyperbolic elements in $\mathbf{PSL}(2,\mathbb{R})$ up to $\mathbf{SL}(2,\mathbb{R})$-conjugation since $\alpha = \exp(-\frac{\ell(A)}{4})$.

Let $A \in \mathbf{SL}(3,\mathbb{R})$ be a positive hyperbolic element. We define a pair of length parameters ℓ and m of A by

$$\ell(A) = \log(\nu/\lambda) \quad \text{and} \quad m(A) = \frac{3}{2}\log(\mu).$$

Then the pair (ℓ, m) is a coordinate in the set of positive hyperbolic elements in $\mathbf{SL}(3,\mathbb{R})$ up to $\mathbf{SL}(3,\mathbb{R})$-conjugation since λ, μ and ν are determined by

$$\lambda = \exp\left(-\frac{\ell(A)}{2} - \frac{m(A)}{3}\right),\ \mu = \exp\left(\frac{\ell(A)}{2} - \frac{m(A)}{3}\right),\ \nu = \exp\left(\frac{2\ m(A)}{3}\right).$$

Therefore the length parameter m measures the deviation of convex real projective structures from hyperbolic structures.

Goldman and Choi [1], [4], [5] showed many properties of the Goldman space $\mathcal{G}(M)$. I give a comprised theorem of them.

Theorem 3.6. *Let M be a compact oriented 2-manifold with $\chi(M) < 0$. Then $\mathcal{G}(M)$ is the* component *of $\mathbb{RP}^2(M)$ containing the Teichmüller space $\mathfrak{T}(M)$ and $\mathcal{G}(M)$ is a real analytic manifold of dimension $-8 \cdot \chi(M)$.*

Let $M = \Sigma(g, 0)$ a closed Riemann surface with genus g and $\chi(M) = 2 - 2g < 0$. Then there exist $3g - 3$ nontrivial homotopically distinct disjoint circles $\{\delta_i\}$ on M such that these circles decompose M as a disjoint union of $2g - 2$ pairs of pants.

Let ℓ_i be the length parameter, θ_i the twisting parameter on the Teichmüller space $\mathfrak{T}(M)$. Then $\{\ell_i, \theta_i \mid i = 1, \ldots, 3g - 3\}$ is a global coordinate on $\mathfrak{T}(M)$ called *Fenchel-Nielsen* coordinates.

Definition 3.7. *A* symplectic *manifold is a smooth manifold M endowed with a nondegenerate closed 2-form ω.*

Theorem 3.8 (Wolpert [11])**.** *Let $M = \Sigma(g,0)$ a closed smooth surface. Then the Teichmüller space $\mathfrak{T}(M)$ is a $6g-6$ dimensional symplectic manifold with the symplectic form*

$$\omega = \sum_{i=1}^{3g-3} d\ell_i \wedge d\theta_i$$

where ℓ_i is the length parameter, θ_i is the twisting parameter on $\mathfrak{T}(M)$.

Let ℓ_i, m_i be length parameters, θ_i, β_i twisting parameters, and s_j, t_j internal parameters on $\mathcal{G}(M)$. Then $\{\ell_i, m_i, \theta_i, \beta_i, s_j, t_j \mid i = 1, \ldots, 3g-3,\ j = 1, \ldots, 2g-2\}$ is a global coordinate on $\mathcal{G}(M)$.

Theorem 3.9 (Kim [6])**.** *Let $M = \Sigma(g,0)$ a closed smooth surface. Then the Goldman space $\mathcal{G}(M)$ is a $16g-16$ dimensional symplectic manifold with the symplectic form*

$$\omega = \sum_{i=1}^{3g-3} d\ell_i \wedge d\theta_i + \sum_{i=1}^{3g-3} dm_i \wedge d\beta_i + \sum_{j=1}^{2g-2} dt_j \wedge ds_j$$

where ℓ_i, m_i are length parameters, θ_i, β_i are twisting parameters, and s_j, t_j are internal parameters on $\mathcal{G}(M)$.

4. An almost Kähler structure on the Goldman space

From now on S is an oriented *closed* Riemann surface with $\chi(S) < 0$. There are many ways to define a symplectic form ω on $\mathcal{G}(S)$. One is we identify $\mathcal{G}(S)$ and the Teichmüller component of $\mathrm{Hom}(\pi, \mathbf{SL}(3,\mathbb{R}))/\mathbf{SL}(3,\mathbb{R})$. The Zariski tangent space to the Teichmüller component at $[\phi]$ is isomorphic to $H^1(S; \mathrm{ad}P_\phi)$ where $\mathrm{ad}P_\phi$ is the flat vector bundle over S with fiber $\mathfrak{sl}(3,\mathbb{R})$ associated to ϕ. Since $H^1(S; \mathrm{ad}P_\phi)$ is isomorphic to de Rham cohomology $H^1_{DR}(S; \mathfrak{sl}(3,\mathbb{R}))$, we may think $\sum_i [\sigma_i \otimes \xi_i]$ is an element of $H^1(S; \mathrm{ad}P_\phi)$ where $\sigma_i \in \Omega^1(S)$ and $\xi_i \in \Gamma(S, \mathfrak{sl}(3,\mathbb{R}))$. Since $\mathfrak{sl}(3,\mathbb{R})$ has a positive-definite symmetric bilinear form B defined by $B(X,Y) = \mathrm{tr}(XY)$, we define a symplectic form ω on $\mathcal{G}(S)$ by

$$\omega([\sigma \otimes \xi], [\sigma' \otimes \xi']) = \int_S (\sigma \wedge \sigma')\ B(\xi, \xi'). \tag{3}$$

We want to define a metric on $\mathcal{G}(S)$. For any $\mathfrak{sl}(3,\mathbb{R})$-valued section ξ, let $\tilde{\xi}$ be the *adjoint section* of ξ defined by $\tilde{\xi}(x) = \xi(x)^t$ for any $x \in S$.

We define a metric $g : H^1(S; \mathrm{ad}P_\phi) \times H^1(S; \mathrm{ad}P_\phi) \to \mathbb{R}$ on $\mathcal{G}(S)$ by

$$g([\sigma \otimes \xi], [\sigma' \otimes \xi']) = \int_S (\sigma \wedge * \sigma')\ B(\xi, \tilde{\xi}') \tag{4}$$

where $* : \Omega^1(S) \to \Omega^1(S)$ is the Hodge star operation. Since each de Rham cohomology class contains a unique harmonic representative, g is a well-defined. The resulting Riemannian metric of $\mathcal{G}(S)$ is called the *Weil-Petersson metric.* For more detail see Darvishzadeh-Goldman [2].

Remark 4.1. Suppose S is a *closed* n-dimensional manifold and $\mathfrak{g}$ has a positive definite symmetric bilinear form. Then the Weil-Petersson metric is still well-defined since the Hodge star operation makes $\sigma \wedge * \sigma'$ n-form and $\sigma \wedge * \sigma' = \sigma' \wedge * \sigma$. But the symplectic form ω is defined if and only if S is a 2-dimensional manifold.

Definition 4.2. An *almost* Kähler manifold (M, J, g, ω) is a smooth manifold M equipped with an almost complex structure J, a Riemannian metric g and a closed nondegenerate skew-symmetric 2-form ω such that

1. $g(X, Y) = g(JX, JY)$
2. $\omega(X, Y) = g(X, JY)$, for any $X, Y \in T_pM,\ p \in M$.

An almost Kähler manifold is called a Kähler manifold if J is integrable.

It is natural to ask whether $\mathcal{G}(S)$ is a Kähler manifold or not. Define an operator J on $\Omega^1(S; \mathrm{ad}P_\phi)$ by

$$J(\sigma \otimes \xi) = - * \sigma \otimes \tilde{\xi} \tag{5}$$

where $* : \Omega^1(S) \to \Omega^1(S)$ is the Hodge star operator and $\tilde{\xi}$ is the adjoint section of ξ. Since $* \circ * = (-1)^{p(n-p)} I$ for $* : \Omega^p(S) \to \Omega^{n-p}$ and $\tilde{\tilde{\xi}} = \xi$,

$$J \circ J(\sigma \otimes \xi) = J(- * \sigma \otimes \tilde{\xi}) = - * (- * \sigma) \otimes \tilde{\tilde{\xi}} = -(\sigma \otimes \xi).$$

Therefore J is an almost complex structure on $\mathcal{G}(S)$ since $J \circ J = -I$.

Theorem 4.3 (Darvishzadeh-Goldman [2])**.** *For above J, g, ω, the Goldman space $\mathcal{G}(S)$ is an almost Kähler manifold.*

Proof. First we will show that $\omega(\alpha, \alpha') = g(\alpha, J\alpha')$ for any $\alpha, \alpha' \in \Omega^1(S; \mathrm{ad}P_\phi)$. Let $\alpha = \sigma \otimes \xi,\ \alpha' = \sigma' \otimes \xi' \in \Omega^1(S; \mathrm{ad}P_\phi)$, then

$$\begin{aligned} g(\alpha, J\alpha') &= g(\sigma \otimes \xi,\ - * \sigma' \otimes \tilde{\xi}') = \int_S \sigma \wedge *(- * \sigma')\ B(\xi, \tilde{\tilde{\xi}}') \\ &= \int_S \sigma \wedge \sigma'\ B(\xi, \xi') = \omega(\alpha, \alpha'). \end{aligned}$$

Since g is symmetric and ω is skew-symmetric,

$$g(J\alpha, J\alpha') = \omega(J\alpha, \alpha') = -\omega(\alpha', J\alpha) = -g(\alpha', JJ\alpha) = g(\alpha', \alpha) = g(\alpha, \alpha').$$

□

Conjecture 4.4 (Goldman)**.** *$\mathcal{G}(S)$ is a Kähler manifold.*

Loftin [8] showed another partial result toward Goldman conjecture. He identify the Goldman space $\mathcal{G}(S)$ and the projectively equivalence classes of torsion free projectively flat connection on S modulo diffeomorphisms which are isotopic to identity map I_S. For a given Riemannian metric, the *conjugate* connection $\bar{\nabla}$ of the torsion free projectively flat connection ∇ with respect to the Levi-Civita connection $\hat{\nabla}$ is defined by $\bar{\nabla} = 2\hat{\nabla} - \nabla$. Then the conjugate connection $\bar{\nabla}$ is also torsion free projectively flat and ∇ is derived from a hyperbolic structure if and only if $\nabla = \bar{\nabla}$.

Theorem 4.5 (Loftin [8])**.** *The Goldman space $G(S)$ has the structure of a holomorphic $5g - 5$ dimensional vector bundle over the Teichmüller space $\mathfrak{T}(S)$ (which has $3g - 3$ complex dimension.) For each fiber $\mathcal{F}$ on a Levi-Civita connection, $\mathcal{F}$ has a Kähler structure.*

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