Trends in Mathematics
Information Center for Mathematical Sciences
Volume 5, Number 1,June 2002, Pages 11–15

ANALYTIC DEFINITIONS FOR HYPERBOLIC OBJECTS

YUNHI CHO AND HYUK KIM

Abstract. We can extend the hyperbolic space beyond the infinity and this extended space which contains hyperbolic space as a subset has many natural properties. The definitions for length, angle, and volume can be given by analytic continuation method. We also obtain theorems for the boundary regularity conditions of 2 and 3 dimensional region with finite volume.

0. Introduction

In [1] and [2] we considered an extended model of hyperbolic space and studied how we can define a volume of a region which lies beyond the infinity of the hyperbolic space. Such investigation gives us a natural way of studying various geometric objects in Lorentz geometry in a manner consistent with those in hyperbolic geometry. The method of calculating volume of such region is essentially an analytic continuation argument and works very well with a region with analytic boundary. But if one tries to consider a region with smooth boundary, the problem turns out to be very delicate.

Now we introduce our main idea and show current results.

Let $\mathbb{R}^{n,1}$ denote the Minkowski space, i.e., $\mathbb{R}^{n+1}$ with the inner product of signature $(n, 1)$ given by

$$< x, y >= -x_0y_0 + x_1y_1 + \cdots + x_ny_n.$$

The hyperbolic space, Lorentz space and the light cone are defined as the sets $\{x \in \mathbb{R}^{n,1} | < x, x >= \alpha\}$ with $\alpha = -1, 1, 0$ respectively together with the induced metric. If we project these sets radially to an affine subspace $\mathbb{K}^n := \{1\}\times\mathbb{R}^n \subset \mathbb{R}^{n,1}$, then we obtain a unit ball as Kleinian projective model for hyperbolic space $\mathbb{H}^n$, Lorentz space of constant sectional curvature 1 outside the ball and the light cone as the common boundary $\partial\mathbb{H}^n$ of these two spaces.

If we change the sign of the induced metric on the Lorentz space, then the new Lorentz space denoted by $\mathbb{L}^n$, has constant sectional curvature -1 and the metrics on both parts $\mathbb{H}^n$ and $\mathbb{L}^n$ have the exactly same formula

$$ds_K^2 = \left(\frac{\Sigma x_i dx_i}{1 - |x|^2}\right)^2 + \frac{\Sigma dx_i^2}{1 - |x|^2}.$$

2000 *Mathematics Subject Classification.* 51M10.

Key words and phrases. Key words and phrases. hyperbolic space, volume form, analytic, Kleinian model.

The second author was supported by grant no.1999-2-101-002-4 from the interdisciplinary research program of the KOSEF.

Received August 30, 2001

And the induced volume form is given by

$$dV_K = \frac{dx_1 \wedge \cdots \wedge dx_n}{(1-|x|^2)^{\frac{n+1}{2}}}.$$

Now for a region U in $\mathbb{H}^n$, the volume of U will be simply given by the integration of dV_K on U. For a region U lying across the boundary of $\mathbb{H}^n$, we formally calculate the volume of U using spherical coordinate as follows:

$$\begin{aligned}\mathrm{vol}(U) &= \int_U \frac{dx_1 \cdots dx_n}{(1-|x|^2)^{\frac{n+1}{2}}} \\ &= \int_{G^{-1}(U)} \frac{r^{n-1}}{(1-r^2)^{\frac{n+1}{2}}} dr d\theta \\ &= \int_a^b \frac{r^{n-1}F(r)}{(1-r^2)^{\frac{n+1}{2}}} dr, \quad F(r) = \int_{G^{-1}(U)} d\theta,\end{aligned}$$

where $G : (r,\theta) \mapsto (x_1, \cdots, x_n)$ is the extended spherical coordinates and $d\theta$ is the volume form of the Euclidean unit sphere $\mathbb{S}^{n-1}$.

Now this integral with respect to r does not make sense in general, but for a region U with analytic boundary we want to use contour integral instead to define a volume of U.

$$\mathrm{vol}(U) := \int_\gamma \frac{r^{n-1}F(r)}{(1-r^2)^{\frac{n+1}{2}}} dr,$$

where γ is a contour given by

$$\gamma(t) = \begin{cases} t, & a \le t \le 1-\delta, \\ 1 + \delta e^{\frac{i(1-t)\pi}{\delta}}, & 1-\delta \le t \le 1, \\ t+\delta, & 1 \le t \le b-\delta. \end{cases} \tag{1}$$

Note that the analyticity of the boundary of U was needed to make sure $F(r)$ is an analytic function of r.

For a region U in Lorentz part, our choice of the contour γ naturally determines the sign of $\mathrm{vol}(U)$ as i^{n+1} and so is determined the sign of dV_K.

In [1], it is shown that $\mathrm{vol}(U)$ can also be obtained through a complex approximation. Let

$$ds_\epsilon^2 = \left(\frac{\Sigma x_i dx_i}{d_\epsilon^2 - |x|^2}\right)^2 + \frac{\Sigma dx_i^2}{d_\epsilon^2 - |x|^2},$$

where $d_\epsilon = 1 - \epsilon i$ with $\epsilon > 0$ and $i = \sqrt{-1}$, so that $ds_K^2 = \lim_{\epsilon\to 0} ds_\epsilon^2$. Then the induced volume form is given by

$$dV_\epsilon = \frac{d_\epsilon dx_1 \wedge \cdots \wedge dx_n}{(d_\epsilon^2 - |x|^2)^{\frac{n+1}{2}}}$$

and let $\mu(U) := \lim_{\epsilon\to 0} \int_U dV_\epsilon$. Here the choice of sign of dV_ϵ is determined by the continuity on $\epsilon \ge 0$ and the sign of dV_K. Then it was shown (see the proposition 1.1 of [1]) that μ is finitely additive and $\mu(U) = \mathrm{vol}\ (U)$ for a region U with an analytic boundary.

The measure theory for μ seems to be very delicate and it is not easy to find a large enough class of μ-measurable sets, that is, Lebesgue measurable sets with $\mu(U) < \infty$ (see [1], [2]).

1. Analytic definitions on the extended model

Already we showed that the volume of the region inside the unit ball in $\mathbb{K}^n$ (i.e., hyperbolic space $\mathbb{H}^n$), $\mathrm{vol}(U) = \int_U dV_K$, can be analytically generalized to the region in $\mathbb{K}^n$ by $\mu(U) = \lim_{\epsilon\to 0}\int_U dV_\epsilon$. Similarly our ϵ-technique can be adapted for other geometric objects, for example, distance between two points, lengths of a curves, angles, k-dimensional volumes and so on (see [1], [3]). The distance between two points on the extended space was first defined by Schlenker [6] by cross ratio. However he considered only distance and angle. For example, the length of γ passing through the ideal boundary is represented by $\lim_{\epsilon\to 0}\int_\gamma ds_\epsilon$ with suitable condition.

From two tangent vectors v_p, w_p at point p on a Riemannian manifold, we can define an angle θ by the equation

$$\langle v_p, w_p\rangle = |v_p||w_p|\cos\theta, \quad 0 \le \theta < \pi.$$

But semi-Riemannian manifold needs a new definition about angle θ, because the function $\cos^{-1}$ is multi-valued and one reasonable choice among them seems to be difficult. In fact, Dzan [5] found the reasonable choice among multi-values through a case by case representation. However we can define the angle $\theta = \angle(v_p, w_p)$ as simple and unique value representation $-i\, d(l_{v_p}\cap p^\perp, l_{w_p}\cap p^\perp)$, where $d(x, y)$ denote the extended distance between x and y, and l_{v_p} is the geodesic line generated from the vector v_p.

The basic properties induced from our definitions are shown at [1], [2], and [3].

2. Flattened model and $C^{1,\alpha}$-boundary region in dimension 3

As we see that 1-dimensional measure gives us length and angle and that k-dimensional measure gives us k-dimensional solid angle, the measure or volume is one of the most fundamental geometric quantity. We discussed the volume of a region with analytic boundary and we want to extend our discussion to a more general type region. And we can determine the optimal regularity condition for the boundary of a region which gives us a meaningful volume.

The computing of integral whose singularities lies on the unit sphere in $\mathbb{K}^n$ is certainly inconvenient and we want to introduce a new model to facilitate the computation. In this model, we want the singularity sets of our volume form as a hyperplane. The immediate choice is a Cayley transformation or a reflection σ with respect to a sphere of radius $\sqrt{2}$ with the center at $e_n = (0, \dots, 0, 1) \in \mathbb{K}^n$.

We see immediately that under the reflection σ, $\mathbb{H}^n$ is sent to the lower half space and $\mathbb{L}^n$ to the upper half space.

From the obvious identities,

$$\begin{cases} y - e_n = \lambda(x - e_n), & \lambda \in \mathbb{R}, \\ |y - e_n||x - e_n| = 2, \end{cases}$$

we easily obtain that $y = \sigma(x)$ is given by

$$\sigma : \begin{cases} y_i = \dfrac{2x_i}{|x - e_n|^2}, & i = 1, \dots, n-1, \\ y_n = \dfrac{2(x_n - 1)}{|x - e_n|^2} + 1. \end{cases}$$

We compute directly using this formula that the metric ds_K^2 is pulled back by σ to

$$ds^2 = \sigma^*(ds_K^2) = \left(\frac{\alpha dx_n - x_n d\alpha}{2\alpha x_n}\right)^2 - \frac{\Sigma dx_i^2}{\alpha x_n},$$

where $\alpha = |x - e_n|^2 = x_1^2 + \cdots + x_{n-1}^2 + (x_n - 1)^2$ so that $d\alpha = 2(\Sigma x_i dx_i - dx_n)$.

Also the volume form dV_K is pulled back to

$$dV = \sigma^*(dV_K) = -\frac{dx_1 \wedge \cdots \wedge dx_n}{2(-x_n)^{\frac{n+1}{2}} \alpha^{\frac{n-1}{2}}}.$$

Here notice that the first negative sign appears since σ is orientation reversing and we can ignore this for computing integrals. If $x_n > 0$, that is, if $x \in \mathbb{L}^n$, we need to determine the sign of $(-1)^{\frac{n+1}{2}}$, and this has been determined as i^{n+1} in the previous section.

This new model $\mathbb{E}^n$ is of course quite different from the Poincare half space model. It is clear from the construction that the geodesics in this model are the circles (including lines viewed as a special case of circles passing through the infinity) passing through the point e_n, and more generally spheres (including planes) passing through e_n are the totally geodesic submanifolds.

Let's consider first the volume of a region U with analytic boundary in the new model $\mathbb{E}^n$. Note that $\sigma^{-1} = \sigma$.

$$\mu(U) = \lim_{\epsilon \to 0} \int_{\sigma(U)} dV_\epsilon = \lim_{\epsilon \to 0} \int_U d\tilde{V_\epsilon},$$

where

$$d\tilde{V_\epsilon} = \sigma^*(dV_\epsilon) = -\frac{1-\epsilon i}{2} \frac{dx_1 \wedge \cdots \wedge dx_n}{(\frac{-\epsilon^2 - 2\epsilon i}{4}\alpha - x_n)^{\frac{n+1}{2}} \alpha^{\frac{n-1}{2}}}.$$

The induced volume form $d\tilde{V_\epsilon}$ has a complicated formula, and instead we use a different simple volume approximation $d\mu_\epsilon$ which gives us the same μ-measure of U.

Theorem 1. *Let U be a region with analytic boundary in $\mathbb{E}^n$ and let*

$$d\mu_\epsilon = \frac{dx_1 \wedge \cdots \wedge dx_n}{2(-x_n - \epsilon i)^{\frac{n+1}{2}} \alpha^{\frac{n-1}{2}}}, \quad \alpha = |x - e_n|^2.$$

Then

$$\mu(U) = \lim_{\epsilon \to 0} \int_U d\mu_\epsilon.$$

Furthermore for a region U with $-\delta < x_n < \delta$,

$$\mu(U) = \int_\gamma \int \frac{dx_1 \wedge \cdots \wedge dx_{n-1}}{2(-x_n)^{\frac{n+1}{2}} \alpha^{\frac{n-1}{2}}} \, dx_n,$$

where γ is a contour from $-\delta$ to δ which goes around 0 clockwise like (1).

Theorem 2. *In the two-dimensional extended hyperbolic space, if a connected region U whose boundary is $C^{0,\frac{1}{2}+\alpha}$-transversal to the $\partial\mathbb{H}^2$, then the area of U is finite and is invariant under hyperbolic isometries.*

Theorem 3. *In the three-dimensional extended hyperbolic space, if a connected region U whose boundary is $C^{1,\alpha}$-transversal to the $\partial\mathbb{H}^3$, then the volume of U is finite and is invariant under hyperbolic isometies.*

The definition of C-transversality and the proofs of the above theorems will be given in [4]. The above boundary regularity condition is in fact sharp and there exists a C^1-transversal region U with infinite volume (see [4]).

References

1. Cho, Y.H. and Kim, H., *Analytic continuation of hyperbolic space*, (preprint).
2. Cho, Y.H., *Finite Additive Measure and Hyperbolic Space*, Trends in Mathematics, **2** (1999), 150-154.
3. Cho, Y.H., *Trigonometry in extended hyperbolic space and extended Lorentz space*, (preprint).
4. Cho, Y.H. and Kim, H., *$C^{1,\alpha}$-boundary region in three dimensional extended hyperbolic space*, (in preparation).
5. Dzan, J.J., *Gauss-Bonnet formula for general Lorentzian surfaces*, Geometriae Dedicata, **15** (1984), 215 - 231.
6. Schlenker, J.M., *Métriques sur les polyèdres hyperboliques convexes*, J.D.G., **48** (1998), 323 - 405.

Yunhi Cho :University of Seoul, Seoul 130-743, Korea
E-mail address: yhcho@uoscc.uos.ac.kr

Hyuk Kim : Seoul National University, Seoul 151-742, Korea
E-mail address: hyukkim@math.snu.ac.kr