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A SURVEY ON FIBRED KNOT AND SYMPLECTIC 4-MANIFOLDS

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Abstract. In this article we reviewed some relations between fibred knot and symplectic 4-manifold. We considered some known results about fibred knots and geometric monodromy of the knot. After that we explained how to construct a symplectic 4-manifold from the information of geometric monodromy as a product of right handed Dehn twists. And then we showed that Stein surface with boundary is equivalent to positive allowable Lefschetz fibration.

1. Introduction

A construction of symplectic 4-manifold and studying its properties is one of main topics in low dimensional topology. It is known that symplectic 4-manifold is characterized by positive Lefschetz fibration [5, 8] and positive Lefschetz fibration can be constructed by using a sequence of right handed Dehn Twists [9].

Fibred knot and link plays a very important role in the construction of symplectic 4-manifold. It is well known that $M^3 \times S^1$ has a symplectic structure if M^3 is fibred over S^1 and $b_1(M^3) \geq 1$. So an easiest way, even though we do not understand clearly about the Lefschetz fibration structure of such a manifold, is to consider the 4-manifold $K_0 \times S^1$ where K_0 is the fibred 3-manifold which is obtained by doing 0-surgery along the fibred knot $K \subset S^3$. The second method is think about a symplectic 4-manifold with boundary which has a geometric monodromy of finite order and take fiber sum to get a Lefschetz fibration over S^2.

Recently, Loi-Piergallini [10] and Akbulut-Ozbagci [1, 2] find some relation between the Stein surface with boundary and Lefschetz fibration. In the paper [10], they found that Stein surfaces with boundary coincides up to orientation preserving diffeomorphisms with simple branched coverings of B^4 whose branch set is a positive braided surface. And a smooth oriented 3-manifold is Stein fillable iff it has a positive open book decomposition.

In this article, we will review some well known properties of 3-manifold topology such as branched covering, open book decomposition, braid, mapping class group based on the papers of Birman [4], Montesinos-Amilibia and Morton [13], Bernstein and Edmonds [3], Rudolphs [14] and then we will explain some new results of Loi-Piergallini [10] and Akbulut-Ozbagci [1, 2].

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2. Symplectic 4-manifold and Lefschetz fibration

Definition 2.1 (Lefschetz fibration)**.** Let X be a smooth oriented compact connected 4-manifold with (possibly empty) boundary and Y be a smooth oriented compact connected (real) surface with (possibly empty) boundary. A proper smooth map $f : X \to Y$ is a *Lefschetz fibration* over Y if

(a) f has finitely many singular values $y_1, y_2, \cdots, y_n \in Int(Y)$ and the restriction of f over $Y - \{y_1, y_2, \cdots, y_n\}$ is a locally trivial fiber bundle whose fiber F is an oriented compact surface with (possibly empty) boundary

(b) for each $i = 1, 2, \cdots, n$, there is only one singular point $x_i \in Int(X)$ over the branch point y_i and the monodromy of a counterclockwise meridian loop around y_i is given by $\delta_i^{\epsilon_i}$, where δ_i is the right-handed Dehn twist along $d_i \in Int(F)$ and $\epsilon_i = \pm 1$ (x_i is called positive or negative depending on ϵ_i).

When we think about a Lefschetz fibration $f : X \to B^2$ with singular values $y_1, y_2, \cdots, y_n$, we can consider it as a sequence of 2-handle attachment to $B^2 \times F$ along the vanishing cycles d_i with framing ϵ_i and the resulting 4-manifold X has boundary $Bd(X) \cong M^3_\varphi$ with global monodromy $\varphi = \delta_n^{\epsilon_n} \circ \delta_{n-1}^{\epsilon_{n-1}} \circ \cdots \circ \delta_1^{\epsilon_1}$ where $M^3_\varphi \cong (F \times I)/_{\sim\varphi}$ when F is closed and M^3_φ is the open book with page F and bind $Bd(F)$ when $Bd(F) \neq \phi$.

We say a Lefschetz fibration $f : X \to Y$ is *positive* if all its singular points x_i are positive and that f is called *allowable* if all the vanishing cycles are homologically nontrivial in F.

Theorem 2.2 ([8], p.286)**.** *A 4-manifold X admits a symplectic structure iff it admits a positive Lefschetz fibration.*

3. Fibred link and its related topics

A fibred link $L \subset S^3$ is a link such that $S^3 - N(L)$ is fibred over S^1 with fiber F. It was known to Harer that

Theorem 3.1. [9] *Let (F, L) be any fibred pair in S^3. Then there are two fibred pairs (F_1, L_1), (F_2, L_2) such that*

(a) *(F_2, L_2) is constructed from the unknot by using Hopf plumbings*

(b) *(F_1, L_1) is obtained from (F, L) using Hopf plumbings*

(c) *(F_1, L_1) may be changed into (F_2, L_2) by using twisting.*

It was conjectured by Harer that any two fibred pairs can be related by using plumbings and deplumbings without any twisting. Stallings [15] showed that any homogeneous braid is a fibred link and the fiber surface which is obtained by using the Seifert algorithm can be obtained by using Hopf plumbings only.

Goldsmith [7] and Birman [4] studied fibred knot by studying the branched covering and braid. After that Montesinos-Amilibia and Morton [13] generalized it as follows

Theorem 3.2. [13] *Let $L \in S^3$ be any fibred link with k components. Then there is a $d = max\{k, 3\}$ fold simple covering $p : S^3 \to S^3$ branched over a closed braid $\hat{\beta} \in B_n$, for some $n \in \mathbb{Z}_+$, with braid axis L_β such that $L = p^{-1}(L_\beta)$ and the fiber $F = p^{-1}(D)$ where D is the disc bounded by L_β. Moreover, we can find a homeomorphism $\phi \in \mathcal{M}(F, Bd(F))$ such that $S^3 - N(L) \cong (F \times I)/\sim\phi$ and ϕ is a lifting of the $\hat{\beta}$ which is considered as a mapping class group of B^2 with n marked points.*

Remark 3.3. [13] If we can show that the closed braid can always be chosen to be the unlink, then Harer's conjecture is true.

A braid is called *completely reducible* if it can be built from the trivial braid on $d-1$ strings by a sequence of Markov moves, each increasing the braid index. In the case each Markov move is related to a Hopf plumbing in the fiber surface. So the related fiber surface can be obtained by a sequence of Hopf plumbings from the disk.

Remark 3.4. (a) When we consider the 3-manifold with boundary $S^3 - N(K)$ for some fibred knot K which has a fiber obtained by a sequence of Hopf plumbings, the geometric monodromy can be obtained easily because one Hopf plumbing is related to a Dehn twist along the center curve of the plumbed Hopf band. The homogeneous braid and especially each torus knot is the case.

(b) When we consider the fibred 3-manifold K_0 comes from the 0-surgery on the given fibred knot K, the fiber is a closed surface of genus $g = \frac{1}{2}deg(\Delta_K(t))$. In the case, we may find the geometric monodromy by using the fact [3]: Any sequence $(\alpha_1, \alpha_2, \cdots, \alpha_k)$ in S_n satisfies (1) $\alpha_1\alpha_2\cdots\alpha_k = id_{S_n}$ (2) Each α_i is a transposition (3) $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ generates a transitive subgroup of S_n, then we can put the sequence in the canonical form $(12), (12), \cdots, (12), (12), (23), (23), (34), (34), \cdots, (n-1\ n), (n-1\ n)$ using the Hurewitz action. So the monodromy may be obtained by using the above theorem and by studying the liftable braid. In the case, we always can get a monodromy as a product of right handed Dehn twists because any left handed Dehn twist can be replaced by a product of right handed Dehn twists in the closed surface case.

4. Compact Stein surface with boundary

In this section, we will review some results given by Loi-Piergallini [10] and Akbulut-Ozbagci [1, 2].

Definition 4.1 (Stein surface)**.** A Stein surface is a non-singular complex surface X which admits a proper strictly plurisubharmonic function $f : X \to [0, +\infty)$ such that BdX is a level set.

Definition 4.2. A smooth oriented closed 3-manifold M is called *Stein fillable* iff it is the oriented boundary of a compact Stein surface X

Definition 4.3 (Braided surface)**.** Let Y and Z be smooth oriented connected compact surfaces. A regularly embedded smooth compact surface $S \subset Y \times Z$ is a *braided surface* over Y iff $\pi_Y|_S : (S, Bd(S)) \to (Y, Bd(Y))$ is a simple branched covering and all branch points lies in $Int(Y)$.

Theorem 4.4. [12] *Each $X^4 = H^0 \cup \lambda H^1 \cup \mu H^2$ is a 3-fold irregular covering space of $B^4 \cong B^2 \times B^2$, the branching set being a braided surface over B^2.*

Now we will understand the 4-manifold X which is a simple branched cover $p : X \to B^2 \times B^2$ branched over a braided surface S over first B^2 as a Lefschetz fibration.

Let $\pi_1 : B^2 \times B^2 \to B^2$ be the projection to the first component. Then $f = \pi_1 \circ p : X \to B^2$ is regular for each regular point on p. We can also check that $x \in X$ is singular iff $p(x)$ is a twist point of S. Let $y_1, y_2, \cdots, y_n \in B^2$ be the branch points of $\pi_1|_S : S \to B^2$. Then $F \cong f^{-1}(y)$ is a branch cover of $\{y\} \times B^2$

branched over $\pi_1^{-1}(y) \cap S$ and $f^{-1}(y_i)$ is a singular fiber which is correspond to the Dehn twist along the simple closed curve which is a lifting of the disk twist about the arc which connecting the two related branch points. We can decide the type of Dehn twist by using the type of the twist point. So we will get the part (a). For the part (b), Fuller [6] showed for the $X - N(F)$ where $X \to S^2$ is a positive allowable Lefschetz fibration with generic fiber F and it can be generalized as followed

Proposition 4.5. [10]

(a) *Let B^2 and Z be smooth oriented connected compact surfaces and let $p : X \to B^2 \times Z$ be a simple branched covering whose branch set is a surface $S \in B^2 \times Z$ braided over B^2. Then $f = \pi_{B^2} \circ p : X \to B^2$ is a Lefschetz fibration.*

(b) *Let $f : X \to B^2$ be an allowable Lefschetz fibration with regular fiber F, If F and $Bd(F)$ are connected, then there is a 3-fold simple covering $p : X \to B^2 \times Z$ branched over a braided surface $S \subset B^2 \times Z$ which is braided over B^2, with $Z \cong S^2$ if F is closed and $Z \cong B^2$ otherwise, such that $f = \pi_{B^2} \circ p$.*

Rudolph [14] proved that : A braided surface $S \subset B^2 \times B^2$ is positive iff it is isotopic to the intersection of a complex analytic curve with $B^2 \times B^2 \subset \mathcal{C}^2$. So if X is PALF over B^2 and X has boundary, then by using the above theorem, X is a branched covering of $B^2 \times B^2$ branched over a complex analytic curve. So $p : X \to B^4$ is an analytic branched covering. So X is a Stein surface.

Eliashberg characterized compact Stein surfaces in terms of handle decompositions.

Theorem 4.6. [8] *A smooth, compact, connected, oriented 4-manifold X admits a Stein structure, up to orientation preserving diffeomorphism, iff it can be presented as a handlebody by attaching 2-handles to a framed link in $Bd(D^4 \cup$ 1-handles$) = \#_m S^1 \times S^2$, where the link is drawn in standard form and the framing coefficient on each link component K is given by $tb(K) - 1$.*

It is also known that

Theorem 4.7 (Lyon, [11])**.** *Let $L \subset S^3$ be a tame link. There exists a torus knot $T \subset S^3$ such that $T \cap L = \phi$ and $L \subset F$, where F is a minimal spanning surface for T.*

From the theorem, Akbulut and Ozbagci [1] observed that for a given Legendrian link diagram in standard form ([8], p.422) L, we can find a torus knot $T(p, q)$ for some $p, q \in \mathbb{Z}_+$ which satisfies the Lyon's Theorem. If we compute the linking number $lk(L_i, L_i^+)$ of the i-th link component by using the orientation of the fiber surface of the torus knot in *square bridge position* , then $tb(L_i) = w(L_i) - \#(\text{right cusps}) = lk(L_i, L_i^+)$.So attaching a 2-handle along a Legendrian link component L_i of L with framing $tb(L_i) - 1$ is same as a right handed Dehn twist along the knot on the fiber F of some torus knot T.

For a torus knot T, $PALF_T$ is diffeomorphic to B^4 because each 1-handle and 2-handle pairs are cancelled. Now when we attach 1-handles, its attaching region is in a neighborhood of the binding T. So attaching a 1-handle extends the fibre surface F to $F \cup H$, a 1-handle attach to the fiber and it does not change the monodromy.

From these observation, we will get

Theorem 4.8. [10, 1] *Given a smooth oriented compact 4-manifold X with boundary, the following statements are equivalent up to orientation preserving diffeomorphisms :*

(a) *X is a Stein surface*
(b) *X is a positive allowable Lefschetz fibration over B^2 with bounded regular fiber.*

It is easy to see that we can consider the Legendrian link L lying on a fiber surface of a infinitely many different torus knot. So for a given compact Stein surface with boundary, we can find infinitely many inequivalent PALF's. [1]

For any open book M_φ, we can find a monodromy $\varphi = \delta_n^{\epsilon_n} \circ \delta_{n-1}^{\epsilon_{n-1}} \circ \cdots \circ \delta_1^{\epsilon_1}$. If the fiber has one boundary components, then we can replace all separating vanishing cycle as a product of non-separating ones. So we get the following

Proposition 4.9. [10] *For any open-book M_φ with page F, there exists a Lefschetz fibration $f : X \to B^2$ with regular fiber F, such that $Bd(X) \cong M_\varphi$. Moreover, we can choose f allowable if $Bd(F)$ is connected and positive if M_φ is a positive open-book.*

From the above Proposition and Theorem 4.8, we will get

Theorem 4.10. [10] *A smooth oriented closed 3-manifold is Stein fillable iff it is orientation preserving diffeomorphic to positive open book.*

Since in the case of closed surface, any monodromy can be written as a product of right handed Dehn twists, $\varphi = \gamma_k \circ \gamma_{k-1} \circ \cdots \circ \gamma_1$ and since $\gamma_k \circ \gamma_{k-1} \circ \cdots \circ \gamma_1 \circ \gamma_1^{-1} \circ \gamma_2^{-1} \circ \cdots \circ \gamma_k^{-1} = 1$, and each γ_i^{-1} can be replaced by a product of right handed Dehn twists, we can find a symplectic 4-manifold $X = X_1 \cup X_2$ where $Bd(X_1) = Bd(X_2)$ is the M_φ. If one of X_i has $b_2^+ = 0$, then by taking fiber sum of symplectic fibration $G(g)$, which comes from the monodromy $(\beta_g \circ \alpha_g \circ \cdots \circ \beta_1 \circ \alpha_1)^{4g+2} = 1$ with generic fiber a Riemann surface of genus g , with X_i we will get

Theorem 4.11. [2] *Any closed surface bundle over S^1 can be embedded into a closed symplectic 4-manifold, splitting the symplectic 4-manifold into two pieces both of which have positive b_2^+.*

As we can see here, lots of interesting theorems in 4-manifold topology can be obtained from some well known facts in 3-manifold topology. It may interest if we can apply some well known facts in symplectic 4-manifolds to knot theory or 3-manifold topology.

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