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REDUCIBILITY THEOREM AND x-FACES

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Abstract. Let M be a simple 3-manifold with a toral boundary component. It is known that if two Dehn fillings on M along the boundary produce reducible manifolds, then the distance between the filling slopes is at most one. This paper gives a remarkably short proof of this result by using x-faces.

1. Introduction

Let M be a compact, connected, orientable 3-manifold with a torus boundary component $\partial_0 M$. Let γ be a *slope* on $\partial_0 M$, that is, the isotopy class of an essential simple closed curve on $\partial_0 M$. The 3-manifold obtained from M by γ-*Dehn filling* is defined to be $M(\gamma) = M \cup V_\gamma$, where V_γ is a solid torus glued to M along $\partial_0 M$ in such a way that γ bounds a meridian disk in V_γ.

By a *small* surface we mean one with non-negative Euler characteristic. We say that a 3-manifold M is *hyperbolic* if M with its boundary tori removed admits a complete hyperbolic structure of finite volume with totally geodesic boundary. Thurston's geometrization theorem for Haken manifolds [Th] asserts that a hyperbolic 3-manifold M with non-empty boundary contains no essential small surfaces. If M is hyperbolic, then the Dehn filling $M(\gamma)$ is also hyperbolic for all but finitely many slopes [Th], and a good deal of attention has been directed towards obtaining a more precise quantification of this statement.

We denote by $\Delta(\gamma_1, \gamma_2)$ the distance, or minimal geometric intersection number, between two slopes γ_1, γ_2 on $\partial_0 M$. In [GL] Gordon and Luecke proved that the distances between two Dehn fillings producing reducing spheres is at most 1. This paper gives a simpler proof of this results.

Theorem 1.1 (Gordon-Luecke)**.** *Suppose that M is hyperbolic. If $M(\gamma_1)$ and $M(\gamma_2)$ contain reducing spheres then $\Delta(\gamma_1, \gamma_2) \leq 1$.*

Hereafter M is a hyperbolic 3-manifold with a torus boundary component $\partial_0 M$. Let $\widehat{F}_i$ be a reducing sphere in $M(\gamma_i)$ with $n_i = |\widehat{F}_i \cap V_{\gamma_i}|$ minimal. Then $F_i = \widehat{F}_i \cap M$ is a punctured sphere properly embedded in M, each of whose n_i boundary components has slope γ_i. Note that $n_1 \geq 3$ because M does not contain essential small surfaces. Note that the minimality of n_i guarantees that F_i is incompressible and ∂-incompressible in M.

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We use i and j to denote 1 or 2, with the convention that, when both appear, $\{i, j\} = \{1, 2\}$. By an isotopy of F_1, we may assume that F_1 intersects F_2 transversely. Let G_i be the graph in $\widehat{F}_i$ obtained by taking as the (fat) vertices the disks $\widehat{F}_i - \mathrm{Int}F_i$ and as edges the arc components of $F_i \cap F_j$ in $\widehat{F}_i$. We number the components of ∂F_i as $1, 2, \cdots, n_i$ in the order in which they appear on $\partial_0 M$. On occasion we will use 0 instead of n_i in short. This gives a numbering of the vertices of G_i. Furthermore it induces a labelling of the end points of edges in G_j in the usual way (see [CGLS]). We can assume that G_1 and G_2 have neither trivial loops nor circle components bounding disks in their punctured spheres, since these spheres are incompressible and boundary-incompressible.

The rest of this section will be devoted to several definitions and well known lemmas. Let x be a label of G_i. An *x-edge* in G_i is an edge with label x at one endpoint, and an *xy*-edge is an edge with label x and y at both endpoints.

An *x-cycle* is a cycle of positive x-edges of G_i which can be oriented so that the tail of each edge has label x. A *Scharlemann cycle* is an x-cycle that bounds a disk face of G_i, only when $n_j \geq 2$. Each edge of a Scharlemann cycle has the same label pair $\{x, x+1\}$, so we refer as $\{x, x+1\}$-Scharlemann cycle. The number of edges in a Scharlemann cycle, σ, is called the *length* of σ. In particular, a Scharlemann cycle of length two is called an *S-cycle* in short.

An *extended S-cycle* is the quadruple $\{e_1, e_2, e_3, e_4\}$ of mutually parallel positive edges in succession and $\{e_2, e_3\}$ form an S-cycle, only when $n_j \geq 4$.

Lemma 1.2. *If G_i contains a Scharlemann cycle, then $\widehat{F}_j$ must be separating, and so n_j is even.*

Proof. Let E be a disk face bounded by a Scharlemann cycle with labels, say, $\{1, 2\}$ in G_i. Let V_{12} be the 1-handle cut from V_{γ_j} by the vertices 1 and 2 of G_j. Then tubing $\widehat{F}_j$ along ∂V_{12} and compressing along E gives such a new $\widehat{F}_j$ in $M(\gamma_j)$ that intersects V_{γ_j} fewer times than the old one. Note that if $\widehat{F}_j$ is non-separating, then the new one is also non-separating, and so reducing. □

Lemma 1.3. *G_i, $i = 1, 2$, cannot contain Scharlemann cycles on distinct label pairs.*

Proof. Without loss of generality assume that σ_1 is a Scharlemann cycle with label pair $\{1, 2\}$ bounding a face E in G_i.

Consider a punctured lens space $L = \mathrm{nhd}((\widehat{F}_j - \mathrm{Int}D) \cup V_{12} \cup E)$ where D is a disk in $\widehat{F}_j$ separated by the edges of σ_1. Since ∂L is a reducing sphere in $M(\gamma_j)$, $2|(\widehat{F}_j - \mathrm{Int}D) \cap V_{\gamma_j}| - 2 = |\partial L \cap V_{\gamma_j}| \geq n_j$ by the minimality of n_j. Thus $|\mathrm{Int}D \cap V_{\gamma_j}| \leq \frac{n_j}{2} - 1$, i.e. the interior of D contains at most $\frac{n_j}{2} - 1$ vertices of G_j.

Suppose there is another Scharlemann cycle σ_2 with distinct label pair. Since the edges of σ_2 lie in the same disk component of $\widehat{F}_j$ separated by the edges of σ_1, the union of the edges of σ_2 and the corresponding two vertices should be contained in a disk containing at most $\frac{n_j}{2}$ vertices (note that σ_1 and σ_2 can share one label). This implies that the other $\frac{n_j}{2}$ vertices of G_j lie in the same disk component of $\widehat{F}_j$ separated by the edges of σ_2, contradicting the same argument of the preceding paragraph. □

Lemma 1.4. *G_i, $i = 1, 2$, cannot contain an extended S-cycle.*

Proof. It follows from [Wu, Lemma 2.3].

In fact one could avoid these reference to [Wu]. Note that an extended S-cycle is the simplest one in x-faces (defined in Section 2) where x is not a label of a Scharlemann cycle. Then these two cases follow immediately from Theorems 2.3. □

The *reduced graph* $\overline{G}_i$ of G_i is defined to be the graph obtained from G_i by amalgamating each family of parallel edges into a single edge.

2. x-face

By Lemma 1.3 we may say that G_i contains only 12-Scharlemann cycles. A disk face of the subgraph of G_i consisting of all the vertices and positive x-edges of G_i is called an *x-face*. Remark that the boundary of an x-face D may be not a circle, that is, ∂D may contain a double edge, and more than two edges of ∂D may be incident to a vertex on ∂D (see Figure 2.1 in [HM]). A cycle in G_i is a *two-cornered cycle* if it is the boundary of a face containing only 01-corners, 23-corners and positive edges, and additionally it contains at least one edge of a 12-Scharlemann cycle. Recall that 0 denotes n_j. A two-cornered cycle must contain both corners and a 03-edge because G_i has only 12-Scharlemann cycles. A *cluster* C is a connected subgraph of G_i satisfying that

(i) C consists of 12-Scharlemann cycles and two-cornered cycles,
(ii) every 12-edge of C belongs to both a Scharlemann cycle and a two-cornered cycle, and
(iii) C contains no cut vertex.

The notions of a two-cornered cycle and a cluster were used firstly in [Ho].

Lemma 2.1. *An x-face, $x \neq 1, 2$, in G_i contains a cluster C.*

Proof. Let Γ_D be the subgraph of G_i in an x-face D. There is a possibility that ∂D is not a circle as mentioned before. Since we will find a pair of two-cornered cycles within D, we can cut formally the graph $G_i \cap D$ along double edges of ∂D and at vertices to which more than two edges of ∂D are incident to so that ∂D is deformed into a circle. (See also Figure 5.1 in [HM].) Thus we may assume that ∂D is a circle and Γ_D has no vertex in the interior of D. We may assume that the labels appear in anticlockwise order around the boundary of each vertex.

Suppose that D has a diagonal edge d with a label pair $\{a, b\}$, which are not of 12-Scharlemann cycles, as in Figure 1(a). Note that these labels must differ from x. Assume without loss of generality that $b < a < x$, i.e. three labels b, a, x appear in anticlockwise order in usual sence. Formally construct a new x-face D' as follows. Keep all corners and edges of Γ_D to the right of d (when d is directed from a to b), discard all corners and edges to the left of d, and then insert additional edges to the left of d, and parallel to d, until you first reach label x at one end of this parallel family of edges, as in Figure 1(b). In particular, these additional edges contain no edges of two-cornered cycles or Scharlemann cycles of the graph on the new x-face D'. Repeat the above process for every diagonal edges which are not of 12-Scharlemann cycles, then we get a new x-face E and a graph Γ_E in E so that all diagonal edges are of 12-Scharlemann cycles and all (and only) boundary edges are x-edges. Furthermore the additional edges contain no edges of Scharlemann cycles or two-cornered cycles of Γ_E. Remark that an xx-edge can appear on the boundary of the graph because we cannot guarantee that $\widehat{F}_j$ is separating.

Figure 1. Split along a diagonal edge

Claim 2.2. Γ_E *contains a* 12*-Scharlemann cycle, so does* Γ_D.

Proof. Assume that Γ_E contains no 12-Scharlemann cycles, and so no diagonal edges. We show first that if for some vertex v of Γ_E two boundary edges are incident to v with label x, then these should be xx-edges. For, in Γ_E at least n_j+1 edges are incident to v. If n_j is even, more than $\frac{n_j}{2}$ mutually parallel edges are incident to v, and so one of these edges should be a yy-edge, $y \neq x$, (recall that Γ_E cannot contain a Scharlemann cycle), a contradiction. If n_j is odd, two families of $\frac{n_j+1}{2}$ mutually parallel edges are incident to v, and the boundary edge of each family should be an xx-edge by the same reason above.

Consider the cycle σ consisting of boundary x-edges of Γ_E. Assume that σ has an x-edge which is not an xx-edge. So only one end has label x at v_1, say. By the fact we just proved, another x-edge incident to v_1 does not have label x at v_1. Thus this edge has label x at the other end v_2, say. After repeating this process, we are led to show that σ is a great x-cycle in the terminology of [CGLS]. (If all boundary edges are xx-edges, still we have a great x-cycle.) By the same argument in the proof of Lemma 2.6.2 of [CGLS], Γ_E contains a Scharlemann cycle, a contradicton. □

This means that $\widehat{F}_j$ must be separating and n_j is even by Lemma 1.2. The parity rule guarantees that each edge of Γ_E connects vertices with one label even and the other label odd, and so there is no xx-edges.

Any 12-edge of a Scharlemann cycle does not belong to $\partial\Gamma_E$. Consider the face E_1 of Γ_E adjacent to the 12-edge which does not bound the Scharlemann cycle. It is possible that E_1 contains more than one 12-edges of Scharlemann cycles. Let $\{a_k, a_k+1\}$, $k = 1, \cdots, n$, be the consecutive label pairs of the corners between two consecutive 12-edges of Scharlemann cycles when runs clockwise around ∂E_1. Note that $a_1 = 2$ and $a_n = 0$.

Assume for contradiction that ∂E_1 is not a two-cornered cycle. Since some a_k then is neither 0 nor 2, there are indices l and m so that $a_k = 0$ or 2 when $1 \leq k < l$ or $k = m$, and $a_k \neq 0, 2$ when $l \leq k < m$.

Consider the edges of the parallelism class containing each $\{a_{k-1}+1, a_k\}$-edge for $l \leq k \leq m$. Since there is no Scharlemann cycles among these edges, one finds that $x \leq a_k < a_{k-1}+1 \leq x$, or $x \leq a_k \leq a_{k-1} < x$. And so $x \leq a_m \leq a_{m-1} \leq$

$\cdots \leq a_l \leq a_{l-1} < x$. This is impossible because $a_{l-1}, a_m = 0$ or 2 and all a_k's are even by the parity rule. Thus ∂E_1 is a two-cornered cycle. Let C be the union of all the Scharlemann cycles and all the two-cornered cycles adjacent to each 12-edges of the Scharlemann cycles. After cutting along cut vertices, a connected component of C is then a desired cluster in Γ_E and so in Γ_D. □

Let $\widetilde{F}_j$ be the twice-punctured sphere obtained from $\widehat{F}_j$ by deleting two fat vertices 1 and 2. The family of all 12-edges of a Scharlemann cycle in the cluster C separates $\widetilde{F}_j$ into disks, and one of those disks contains both vertices 0 and 3 of G_j because of the existence of 03-edges in C. The two 12-edges bounding such a disk is called *good edges* of C. Thus each Scharlemann cycle in C has exactly two good edges.

Let Λ be the maximal dual graph of C whose vertices are dual to Scharlemann cycles and two-cornered cycles containing good edges, and edges are dual to good edges of C as depicted in Figure 2. Thus in Λ, a vertex dual to a Scharlemann cycle has valency 2, and a vertex dual to a two-cornered cycle has valency the number of good edges of the two-cornered cycle. Furthermore Λ is a tree according to the construction of C. This implies that each 12-edge, which is not a good one, of two-cornered cycles related to vertices of Λ contributes to the number of components of Λ by adding 1. Consequently there is a component Λ_g of Λ so that all 12-edges of its dual two-cornered cycles are good. Hereafter, we consider the subgraph C_g

FIGURE 2. A cluster and a seemly pair

of C, dual to Λ_g. Say, C_g contains n Scharlemann cycles, and so $2n$ good edges and $n+1$ two-cornered cycles. A two-cornered cycle dual to an end vertex of the tree Λ_g has only one good edge. Choose one e_1 of the nearest edges to vertex 0 (or 3) among them, i.e. there is no such good edges between e_1 and vertex 0 in $\widetilde{F}_j$. Let σ_g and σ_1 be the Scharlemann cycle and two-cornered cycle adjacent to e_1 respectively. Then σ_g has another good edge e_2, and e_1 and e_2 bound a disk D_g containing 0 and 3 in $\widetilde{F}_j$. Note that all Scharlemann cycles are parallel on a torus obtained from $\widetilde{F}_j$ by attaching an annulus ∂V_{12}. Thus exactly $n-1$ out of $2n$ good edges are not contained in D_g. Therefore we have another two-cornered cycle σ_2 all of whose 12-edges lie in D_g. Consequently all edges (consisting of 01-edges, 12-edges, 23-edges and 03-edges) of σ_1 and σ_2 lie in D_g. Furthermore if σ_2 has only one good 12-edge, then the two good edges of σ_1 and σ_2 lie on different sides of the vertices 0 and 3 in D_g. Such σ_1, σ_2 are called a *seemly pair*.

From now we apply the argument in [Ho, Section 6] to get the following two theorems.

Theorem 2.3. *G_i, $i = 1, 2$, cannot contain an x-face for $x \neq 1$ or 2.*

Proof. Suppose that G_i contains such an x-face. We continue the preceding argument. Recall that $|\text{Int}D_g \cap V_{\gamma_j}| \leq \frac{n_j}{2} - 1$ as in the proof of Lemma 1.3. Let $X'_D = \text{nhd}(D_g \cup V_{01} \cup V_{23})$ and $X'_F = \text{nhd}(\widehat{F}_j \cup V_{01} \cup V_{23})$. Then X'_D is a genus two handlebody and X'_F is a once-punctured genus two handlebody. The genus two torus component of $\partial X'_F$ is referred to as the outer boundary of X'_F. Let E_i be the face bounded by the two-cornered cycle σ_i for $i = 1, 2$. The point is that all edges of σ_i are contained in D_g. Let $X_D = X'_D \cup \text{nhd}(E_1) \cup \text{nhd}(E_2)$ and $X_F = X'_F \cup \text{nhd}(E_1) \cup \text{nhd}(E_2)$ as in Figure 3. Since σ_1 is non-separating on $\partial X'_D$ and $\partial X'_F$, both ∂X_D and the outer components of ∂X_F are either a 2-sphere or the disjoint union of a 2-sphere and a torus, simultaneously. Note that the latter case occurs only when σ_1 and σ_2 are parallel. First, assume that ∂X_D is a 2-sphere S_D,

FIGURE 3. X_D and X_F

and the outer component of ∂X_F is a 2-sphere S_F. If X_D is not a 3-ball, then S_D is a reducing sphere. By the previous remark $|S_D \cap V_{\gamma_j}| = 2|D_1 \cap V_{\gamma_j}| \leq n_j - 2$, contradicting the minimality of n_j. Thus it should be a 3-ball, and so X_F is homeomorphic to $S^2 \times I$. Thus S_F is isotopic to $\widehat{F}_j$, contradicting the minimality of n_j again.

Next, assume that ∂X_D is the disjoint union of a 2-sphere and a torus, $S_D \cup T_D$ and the outer components of ∂X_F are also the disjoint union of a 2-sphere and a torus, $S_F \cup T_F$. Recall that this case occurs whenever σ_1 and σ_2 cobound an annulus in $\partial X'_D$ and $\partial X'_F$. This is possible only when σ_2 corresponds to an end vertex of Λ_g with the same number of 01-corners and 23-corners as that of σ_1. Let

D' be the intersection of ∂X_D and the inner sphere component of ∂X_F. By the choice of the seemly pair, this annulus in $\partial X'_D$ contains D', so does S_D. Similarly S_F contains a pushoff of $\widetilde{F}_j - D_g$.

If S_D is non-separating in $M(\gamma_j)$, then it is a reducing sphere with $|S_D \cap V_{\gamma_j}| \leq n_j - 2$, contradicting. Thus S_D, and hence T_D are separating in $M(\gamma_j)$. Let X''_D be the manifold bounded by S_D containing T_D in $M(\gamma_j)$. If X''_D is not a 3-ball, then S_D is a reducing sphere with less intersection with V_{γ_j}. Thus it should be a 3-ball, and so S_F is isotopic to $\widehat{F}_j$, contradicting again. This completes the proof. □

3. Proof of Theorem 1.1

In this section we prove the main theorem. Assume for contradiction that $\Delta \geq 2$.

If a vertex x of G_1 has more than $n_2 - 2$ negative edges, then G_2 contains more than $n_2 - 2$ positive x-edges by the parity rule. Thus the subgraph Γ_x of G_2 consisting of all vertices and positive x-edges of G_2 has n_2 vertices and more than $n_2 - 2$ edges. Then an Euler characteristic calculation shows that Γ_x contains a disk face, that is, an x-face.

Thus by Theorem 2.3 each vertex $x \neq 1, 2$, the labels of Scharlemann cycles of G_2 if it exist, has at least $(\Delta - 1)n_2 + 2$ positive edges. Let G_1^+ denote the subgraph of G_1 consisting of all vertices and positive edges of G_1. Let Λ' be an extremal component of G_1^+. That is, Λ' is a component of G_1^+ having a disk support D such that $D \cap G_1^+ = \Lambda'$. In Λ', a block Λ with at most one cut vertex is called an *extremal block*. If Λ' has no cut vertex, then Λ' itself is an extremal block, and if Λ' has a cut vertex, then it has at least two extremal blocks.

Furthermore, if G_2 contains a 12-Scharlemann cycle, then $\widehat{F}_1$ is separating, and so G_1^+ is disconnected. Then there is an extremal component of G_1^+ which contains at most one of vertices 1 and 2. Now choose an extremal block so that it contains at most one such vertex or a cut vertex which is called a *ghost vertex* y_0.

Therefore G_1 contains an extremal block Λ, each of whose boundary vertex except y_0 has at least $n_2 + 2$ consecutive edge endpoints, and so has all different n_2 labels.

If there is no ghost y_0, choose any label x' but 1 and 2, the labels of Scharlemann cycles of G_1. Or if such y_0 exists and it has more than two edges incident there, then choose x' among the labels of y_0 except 1 and 2. Let $\Gamma_{x'}$ be the subgraph of Λ consisting of all vertices and x'-edges. Then in $\Gamma_{x'}$ the number of edges cannot be less than the number of vertices. Again an Euler characteristic calculation of $\Gamma_{x'}$ on the disk guarantees the existence of x'-face, $x' \neq 1, 2$, contradicting Theorem 2.3. For the remaining case, if y_0 has only two edges incident there, then delete y_0 and the two edges from Λ. Since all labels still appear on each vertex of this new Λ, we can proceed the same argument above to get a contradiction.

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