Trends in Mathematics
Information Center for Mathematical Sciences
Volume 5, Number 1,June 2002, Pages 47–51

FLUX ON COMPACT SYMPLECTIC-CONTACT MANIFOLDS

MYUNG IM LIM

Abstract. We introduce flux homomorphism and flux group on a closed symplectic manifold. And on a compact symplectic-contact manifold $(M, \omega, \partial M, \alpha)$ we construct a flux homomorphism and investigate the flux group.

1. Introduction

In this section, we introduce basic definitions and the flux conjecture. We begin with a closed manifold (M, ω). Let $\mathrm{Symp}_0(M, \omega)$ be the identity component of the group of the symplectic diffeomorphisms of M, and $\widetilde{\mathrm{Symp}}_0(M, \omega)$ the universal cover of $\mathrm{Symp}_0(M)$. A point in $\widetilde{\mathrm{Symp}}_0(M)$ is a homotopy class of smooth paths $\phi_t \in \mathrm{Symp}_0(M)$ with fixed endpoints $\phi_0 = id$ and $\phi_1 = \phi$. Denote by $\{\phi_t\}$ such a homotopy class. Then the flux homomorphism $F : \widetilde{\mathrm{Symp}}_0(M) \to H^1(M; \mathbb{R})$ is defined by

$$F(\{\phi_t\}) = \int_0^1 [\iota(X_t)\omega] dt,$$

where the symplectic vector field X_t satisfies $X_t \circ \phi_t = \frac{d}{dt}\phi_t$ and $\iota(X_t)\omega$ is the contraction of of ω by the X_t. This homomorphism depends only on the homotopy class of the path ϕ_t. Recall that a path is Hamiltonian if its generating family $\iota(X_t)\omega$ is exact at each time t. Let us denote by $\mathrm{Ham}(M)$ the group of Hamiltonian diffeomorphisms, that is the one of elements of $\mathrm{Symp}_0(M)$ which are endpoints of Hamiltonian paths and by $\widetilde{\mathrm{Ham}}(M)$ the universal cover of $\mathrm{Ham}(M)$. We define the flux group Γ of $H^1(M; \mathbb{R})$ to be the image by F of loops in $\mathrm{Symp}_0(M)$:

$$\Gamma = \mathrm{Image}\{F : \pi_1(\mathrm{Symp}_0(M)) \to H^1(M; \mathbb{R})\}.$$

Theorem 1.1 [M.S]. *Let $\{\phi_t\}$ be a path of symplectic diffeomorphisms. Then $\{\phi_t\}$ is isotopic to a Hamiltonian isotopy with fixed endpoints if and only if $F(\{\phi_t\}) = 0$.*

Theorem 1.2. *Let $\{\phi_t\}$ be a path of symplectic diffeomorphisms. Then the endpoint $\phi_1 = \phi$ is a Hamiltonian diffeomorphism if and only if $F(\{\phi_t\}) \in \Gamma$.*

Proof. First we suppose that the end point $\phi_1 = \phi$ is an element in $\mathrm{Ham}(M)$ for given path $\{\phi_t\}$ of symplectic diffeomorphisms. Thus there exists a path $\{\psi_t\}$ of

2000 *Mathematics Subject Classification.* 37K65, 53D05.
Key words and phrases. Hamiltonian Diffeomorphism, symplectic manifold.
Received August 30, 2001

Hamiltonian diffeomorphisms from $\psi_0 = \text{id.}$ to $\psi_1 = \phi = \phi_1$. From Theorem 1.1 we can obtain a loop $\{\psi_t^{-1} \circ \phi_t\} \in \pi_1(\text{Symp}(M))$ satisfying

$$\begin{aligned} F(\{\psi_t^{-1} \circ \phi_t\}) &= F(\{\psi_t^{-1}\}) + F(\{\phi_t\}) \\ &= F(\{\phi_t\}). \end{aligned}$$

The second equality induced from the fact that $\{\psi_t^{-1}\}$ is a path of Hamiltonian diffeomorphisms. This proves that $F(\{\phi_t\}) \in \Gamma$. Inversely, we assume $F(\{\phi_t\}) \in \Gamma$. Then there exists a loop $\{\psi_t\}$ satisfying $F(\{\phi_t\}) = F(\{\psi_t\})$. Thus we can find a loop $\{{\phi'}_t^{-1} \circ \phi_t\}$ is homotopic to $\{\psi_t\}$, where $\{{\phi'}_t\}$ is a path from ${\phi'}_0 = \text{id.}$ to ${\phi'}_1 = \phi = \phi_1$. From this

$$\begin{aligned} F(\{\phi_t\}) &= F\{\psi_t\} \\ &= F(\{{\phi'}_t^{-1} \circ \phi_t\}) \\ &= F(\{{\phi'}_t^{-1}\}) + F(\{\phi_t\}) \end{aligned}$$

and hence $F(\{{\phi'}_t^{-1}\}) = 0$. By Theorem1.1 $\{{\phi'}_t\}$ is a path of Hamiltonian diffeomorhisms and the end point ${\phi'}_1 = \phi = \phi_1$ is a Hamiltonian diffeomorphism. □

From Theorems 1.1 and 1.2 there are exact sequences:

1. $0 \to \widetilde{\text{Ham}(M)} \to \widetilde{\text{Symp}_0(M)} \to H^1(M;\mathbb{R}) \to 0$,
2. $0 \to \text{Ham}(M) \to \text{Symp}_0(M) \to H^1(M;\mathbb{R})/\Gamma \to 0$.

Remark 1.3 [M.S]. The above concepts make sense for noncompact manifolds without boundary. So there exists an exact sequence under compact-open topology:

$$0 \to \text{Ham}(M) \to \text{Symp}_0(M) \to H^1(M;\mathbb{R})/\Gamma \to 0$$

Theorem 1.4 [L.M.P1]. *For any closed symplectic manifold, Γ is discrete if and only if the subgroup of Hamiltonian diffeomorphisms is C^1-closed in the full group of (symplectic) diffeomorphisms of the manifold.*

We can now state the main conjecture as following two equivalent statements.

Flux Conjecture.

1. *For any symplectic manifold, Ham(M) is C^1-closed in Symp$_0(M)$.*
2. *For any closed symplectic manifold, the flux group Γ is discrete.*

2. Flux homomorphism on compact symplectic-contact manifolds

In this section we would like to introduce compact symplectic-contact manifolds and construct the flux homomorphism on compact symplectic-contact manifolds using the flux homomorphim on the noncompact manifolds which is well defined. Let (M, ω, C, α) be a symplectic-contact manifold such that (M, ω) is an open symplectic manifold with compactification $\bar{M} = M \cup C$ and (C, α) is a contact manifold with a contact 1-form α satisfying $d\alpha = \omega|_C$. A symplectic-contact manifold (M, ω, C, α) is compact if $C = \partial M$. This means that (M, ω) is a symplectic manifold with contact type boundary $(\partial M, \alpha)$. We denote by $(M, \omega, \partial M, \alpha)$ a compact symplectic-contact manifold. From now on we consider a symplectic-contact manifold as a compact one when no specific mention is made.

Lemma 2.1[D.S]. *Let (C, α) be a contact manifold. Then $\psi : C \to C$ is a contactomorphism with $\psi^*\alpha = \alpha$ if and only if $\phi : C\times\mathbb{R} \to C\times\mathbb{R}$ is a symplectomorphism on $(C\times\mathbb{R}, d(e^s\alpha))$ where ϕ is defined as $\phi(m, s) = (\psi(m), s)$ for $q = (m, s) \in C\times\mathbb{R}$.*

Theorem 2.2. *For any compact symplectic-contact manifold $(M, \omega, \partial M, \alpha)$, there exists a symplectic form $\tilde{\omega}$ on the noncompact manifold $(\tilde{M}, \tilde{\omega}) = (M \cup \partial M \times (-\varepsilon, \infty), \tilde{\omega})$, which is defined by*

$$\tilde{\omega}_q = \begin{cases} \omega_q & \text{if } q \in M, \\ d(e^s\alpha) = e^s(d\alpha - \alpha \wedge ds) & \text{if } q = (m, s) \in \partial M \times [0, \infty). \end{cases}$$

Proof. Consider an embedding $\Phi : \partial M \times (-\varepsilon, 0] \to M$ of codimension 0 such that

$$\Phi(m, 0) = m \in \partial M \quad \text{on} \quad (m, 0) \in \partial M \times \{0\} \quad \text{and}$$
$$\Phi^*\omega(m, s) = d(e^s\alpha)(m, s) \quad \text{on } (m, s) \in \partial M \times (-\varepsilon, 0].$$

If $q = (m, s) \in \partial M \times \{0\} = \partial M$, since $\frac{\partial}{\partial s}$ is transverse to $\partial M = \partial M \times \{0\}$,

$$\begin{aligned} \Phi^*\omega(m, s)|_{\partial M\times\{0\}} &= d(e^s\alpha)|_{\partial M\times\{0\}} \\ &= e^s(d\alpha - \alpha \wedge ds)|_{\partial M\times\{0\}} \\ &= d\alpha. \end{aligned}$$

Gluing (M, ω) and $(\partial M\times(-\varepsilon, \infty), d(e^s\alpha))$ by the map Φ, we can obtain a symplectic manifold $(\tilde{M}, \tilde{\omega}) = (M \cup \partial M \times (-\varepsilon, \infty), \tilde{\omega})$ as follows.

We identify a neighborhood of ∂M in M with $\partial M \times (-\varepsilon, 0]$ in $\partial M \times (-\varepsilon, \infty)$ using Φ. Choosing ε' small enough , we can make $\phi_t[\Phi(\partial M \times (-\varepsilon', 0])] \subset \text{Image}(\Phi)$ for all $t \in [0, 1]$. Then $\Phi^{-1} \circ \phi_t \circ \Phi : \partial M \times (-\varepsilon', 0] \to \partial M \times (-\varepsilon, 0] \subset \partial M \times (-\varepsilon, \infty)$ is an embedding of codimension 0. On $\partial M = \partial M \times \{0\}$

$$\Phi^{-1} \circ \phi_t \circ \Phi(m, 0) = \Phi^{-1} \circ \phi_t|_{\partial M}(m) = \Phi_0^{-1}(\psi_t(m)) = (\psi_t(m), 0),$$

where $\psi_t = \phi_t|_{\partial M} : \partial M \to \partial M$ is a contactomorphism and $\Phi_s(m) := \Phi|_{\partial M\times\{s\}}(m)$. Then $\tilde{M}$ is the manifold that we desired. □

Theorem 2.3. *Let $\{\phi_t\}$ be a path in $Symp_0(M)$. Then $\{\phi_t\}$ can be extended to a homotopy equivalent path $\{\tilde{\phi}_t\}$ in $Symp_0(\tilde{M})$.*

Proof. Define $\tilde{\phi}_t : \tilde{M} \to \tilde{M}$ by

$$\tilde{\phi}_t(q) = \begin{cases} \phi_t(q) & \text{if } q \in M, \\ (\psi_t(m), s) & \text{if } q = (m, s) \in \partial M \times [0, \infty), \end{cases}$$

where $\psi_t(m) = \phi_t|_{\partial M}(m)$ is a contactomorphism on ∂M. From Lemma 2.1 we find that $\tilde{\phi}_t|_{\partial M\times[1,\infty)} = (\psi_t(m), s)$ is a symplectomorphism and $\tilde{\phi}_t^*\tilde{\omega}|_M = \phi_t^*\omega = \omega$. Then $\tilde{\phi}$ is a well-defined symplectomorphism. □

We denote $X_t \circ \phi_t = \frac{d}{dt}\phi_t$, $\chi_t \circ \psi_t = \frac{d}{dt}\psi_t$. Then we have

$$\frac{d}{dt}\tilde{\phi}_t(q) = \begin{cases} X_t \circ \phi_t(q) & \text{if } q \in M, \\ (\chi_t \circ \psi_t(m), 0) & \text{if } q = (m, s) \in \partial M \times [0, \infty). \end{cases}$$

Thus we have

$$\iota(\tilde{X}_t)\tilde{\omega}_q = \begin{cases} \iota(X_t)\omega_q & \text{if } q \in M, \\ \iota(\chi_t(m))d(e^s\alpha) & \text{if } q = (m,s) \in \partial M \times [0,\infty). \end{cases} \quad (2.1)$$

Recall that the flux homomorphim on the noncompact manifold $(\tilde{M},\tilde{\omega})$ is well defined(Remark 1.3) and that the inclusion $i : M \to \tilde{M}$ induces the isomorphism $i^* : H^1(\tilde{M},\mathbb{R}) \to H^1(M,\mathbb{R})$. Using the extension $\{\tilde{\phi_t}\}$ of the path $\{\phi_t\}$, we can define the flux homomorphim on compact symplectic-contact manifolds M, $F : \widetilde{\mathrm{Symp}_0}(M) \to H^1(M;\mathbb{R})$, as $F(\{\phi_t\}) := i^*(\int_0^1 [\iota(\tilde{X}_t)\tilde{\omega}]dt)$.

3. Flux Conjecture on compact symplectic-contact manifolds

We would like to investigate flux group Γ on compact symplectic-contact manifolds and prove the flux conjecture with some ristricted condition. We can check easily that Theorem 1.1 and Theorem 1.2 hold for compact symplectic-contact manifolds. The Theorems in this section are to be proved in another paper of mine.

Theorem 3.1. *Let M be a compact symplectic-contact manifold and $F : \widetilde{Symp_0}(M) \to H^1(M;\mathbb{R})$ is the flux homomorphism. Then there exists a right inverse $s : H^1(M;\mathbb{R}) \to \widetilde{Symp_0}(M)$ of F such that $F \circ s = id$.*

From this theorem we get the surjectivity of flux homomorphism on symplectic-contact manifolds. Thus there exist exact sequences for a compact symplectic-contact manifold $(M,\omega,\partial M,\alpha)$:

1. $0 \to \widetilde{\mathrm{Ham}}(M) \to \widetilde{\mathrm{Symp}_0}(M) \to H^1(M;\mathbb{R}) \to 0$
2. $0 \to \mathrm{Ham}(M) \to \mathrm{Symp}_0(M) \to H^1(M;\mathbb{R})/\Gamma \to 0$,

and Theorem 1.4 can be extended to a compact case as follow.

Theorem 3.2. *For any compact symplectic manifold which may have a contact boundary, Γ is discrete if and only if the subgroup of Hamiltonian diffeomorphisms is C^1-closed in the full group of (symplectic) diffeomorphisms of the manifold.*

Thus we can state the flux conjecture for a compact symplectic-contact manifold as folowing two equivalent statements:

Flux Conjecture.
1. *For a compact symplectic-contact manifold, $Ham(M)$ is C^1-closed in $Symp_0(M)$.*
2. *For a compact symplectic-contact manifold, The flux group Γ is discrete.*

Indeed, it is suffices to check the discreteness of the flux group Γ in order to prove the flux conjecture.

Theorem 3.2. *Let M be a compact symplectic-contact manifold. Then the flux conjecture holds for the following cases:*
1. *M is a Lefschetz manifold and $H^{2n-2}(\partial M) = 0$.*
2. *M has a symplectic torus action.*
3. *M is spherically rational.*
4. *The first betti number equals to 1.*

References

[B] Glen E. Bredon, *Topology and Geometry*, Springer-Verlag New York, Inc. (1993).

[G.H] P.Griffiths and J.Harris, *Principles of algebraic geometry*, John Wiley and Sons, New York (1978).

[L.M.P1] F.Lalonde, D.McDuff, and L.Porterovich, *On the Flux Conjectures*, preprint.

[L.M.P2] F.Lalonde, D.McDuff, and L.Porterovich, *Topological rigidity of Hamiltonian loops and Quantum homology*, preprint.

[McD] D.McDuff, *Symplectic diffeomorphisms and the flux homomorphism*, Invent. Math. 77, 353-366 (1984).

[M.S] D.McDuff and D.Salamon, *Introduction to symplectic topology*, Clarends press, Oxford (1995).

DEPARTMENT OF MATHEMATICS, EWHA WOMEN'S UNIVERSITY, SEOUL 120-750, KOREA
E-mail address: julyinseoul@hanmail.net