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RIGIDITY ON SYMMETRIC SPACES

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Abstract. In this note we survey rigidity results in symmetric spaces. More precisely, if X and Y are symmetric spaces of noncompact type without Euclidean de Rham factor, with G_1 and G_2 corresponding semisimple Lie groups, and $\Gamma_1 \subset G_1, \Gamma_2 \subset G_2$ are Zariski dense subgroups with the same marked length spectrum, then $X = Y$ and Γ_1, Γ_2 are conjugate by an isometry. As an application, we answer in the affirmative a Margulis's question and show that the cross-ratio on the limit set determines the Zariski dense subgroups up to conjugacy. We also embeds the space of nonparabolic representations from Γ to G into $\mathbb{R}^\Gamma$. Other applications are about projective structures on manifolds equipped with the Hilbert metric.

1. Introduction

In this note, we survey the rigidity of symmetric spaces in various view points. We want to look at the symmetric spaces in terms of dynamics and Lie group theory tied with geometry. For Riemannian and representation theoretical point of view, we present the marked length rigidity of Zariski dense subgroups of the semisimple Lie groups. For dynamical point of view, we give some results about geodesic flows and cross-ratio on the ideal boundary. As applications, we give rigidity result on projective manifolds and prove an analogue of Hitchin's conjecture in dimension 3.

2. Symmetric spaces of noncompact type

The best way to describe a symmetric space of noncompact type is through the language of Lie group. Any symmetric space can be decomposed into the product of a Euclidean, a compact type and a noncompact type symmetric space. In this paper we are concerned with a noncompact type symmetric space, equivalently a symmetric space with nonpositive sectional curvature without Euclidean factor. Such a symmetric space X can be identified with G/K where G is a semisimple Lie group of noncompact type with trivial center and K is a maximal compact subgroup. The group G is identified with the identity component of the isometry group of X and K is identified with an isotropy group of some point x_0 in it. In general G is decomposed into a direct product of noncompact simple groups G_i without center and X into a product of irreducible symmetric spaces X_i. Each G_i is a normal factor of G and only normal subgroups of G are the products of G_i.

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Denote the Lie algebra of G (respectively K) by $\mathfrak{g}$ (respectively $\mathfrak{k}$). Then there is a direct sum decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

with a Cartan relation $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. The killing form B defined by

$$B(Y, Z) = Tr(adY \circ adZ)$$

is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. Identifying $\mathfrak{p}$ with a tangent space of X at a point x_0, one obtains a G-invariant Riemannian metric on X. The sectional curvature is given by $K(Y, Z) = -||[Y, Z]||^2$. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$. One calls the dimension of $\mathfrak{a}$ the **rank** of the symmetric space X. Note $(\exp \mathfrak{a})x_0$ is a maximal flat in X through $x_0 \in X$.

By the standard theory of Lie algebra, one obtains the root space decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Lambda} \mathfrak{g}_\alpha.$$

The set $\mathfrak{a}_{sing} = \{H \in \mathfrak{a} | \exists \alpha \in \Lambda, \alpha(H) = 0\}$ of singular vectors divides $\mathfrak{a}$ into the finite number of components called **Weyl chambers**. Fixing a component $\mathfrak{a}^+$ amounts to choosing positive roots

$$\Lambda^+ = \{\alpha \in \Lambda | \alpha(H) > 0, \forall H \in \mathfrak{a}^+\}.$$

Furthermore if one sets

$$\mathfrak{n}^\pm = \sum_{\alpha \in \Lambda^\pm} \mathfrak{g}_\alpha$$

one obtains the **Iwasawa** decomposition KAN where $N = \exp \mathfrak{n}^+$. A fundamental system $\Pi \subset \Lambda^+$ is a basis of the root system such that any element in Λ is an integral linear combination of the elements in Π with the coefficients the same sign.

A **geometric boundary** (or ideal boundary) ∂X of X is defined as the set of equivalence classes of geodesic rays under the equivalence relation that two rays are equivalent if they are within finite Hausdorff distance each other. For any point $x \in X$ and any ideal point $\xi \in \partial X$, there exists a unique ray $\overline{x\xi}$ starting from x which represents ξ.

3. Marked length rigidity

Partial results of the following theorem are published in various places [10, 11, 3].

Theorem 1. *Let X and Y be symmetric spaces of noncompact type such that G_1 and G_2 are corresponding semisimple Lie groups with trivial center. Let $\Gamma_1 \subset G_1$ and $\Gamma_2 \subset G_2$ be Zariski dense subgroups such that there is a surjective homomorphism $\phi : \Gamma_1 \rightarrow \Gamma_2$ which preserves the translation lengths. Then X and Y are isometric, and Γ_1, Γ_2 are conjugate.*

The following is proved in [1] corollary 3.5.

Lemma 1. *Suppose $g_1, \cdots, g_l$ are Π-proximal elements (i.e. hyperbolic elements) in a higher rank semisimple Lie group of noncompact type. Then*

$$\lim_{n\rightarrow\infty} \frac{\lambda(g_1^n \cdots g_l^n)}{\lambda(g_1)^n \cdots \lambda(g_l)^n} = \nu(g_1, \cdots, g_l).$$

Definition 1. *Let X be a symmetric space of noncompact type without Euclidean factor with $Iso^0(X) = G$. Let x_1, x_2, x_3, x_4 be four points in $X \cup \partial X$ so that x_i and x_j, where $\{ij\} = \{14\}, \{13\}, \{23\}, \{24\}$, can be joined by geodesics l_{ij} respectively. Let v_{ij}^n be geodesic segments approximating l_{ij} and oriented from i to j. Then v_{ij}^n lies in a maximal flat. Identify this maximal flat with A ($G = KAN$) such that the origin of A is equal to the starting point of v_{ij}^n. Let a_{ij}^n be a unique conjugate of v_{ij}^n in a fixed Weyl chamber A^+ under this identification. Then the cross ratio is defined by*

$$[x_1, x_2, x_3, x_4] = \lim_{n\to\infty} \frac{a_{13}^n a_{24}^n}{a_{14}^n a_{23}^n} \in A^+.$$

It is not difficult to see that the limit exists and it is independent of the choices involved using horosphere argument as in [10].

Let g, h be regular hyperbolic (Π-proximal) elements in X. Then g and h have attracting and repelling fixed points in ∂X, denoted by $g^\pm, h^\pm$ respectively. Note in this case, $\lambda(g^n), \lambda(h^n)$ play the role of a_{13}^n, a_{24}^n respectively in our definition of the cross-ratio $[g^-, h^-, g^+, h^+]$. Then as Lemma 1 shows, $\lambda(g^n h^n)$ plays the role of $a_{14}^n a_{23}^n$. When the rank of a symmetric space is one, Lemma 1 is proved in [10]. Then as in [10]

Lemma 2. *The cross-ratio on $X \cup \partial X$ satisfies*

1. $[a, b, x, y][b, c, x, y] = [a, c, x, y],\ [a, b, x, y][a, b, y, z] = [a, b, x, z]$
2. $[g^-, g^+, \eta, g(\eta)] = \lambda(g)^{-2}$ *where η is any point in ∂X such that $g^n(\eta)\to g^+$*
3. $[g^-, h^-, g^+, h^+] = \nu(g, h)^{-1}$ *for any hyperbolic isometries g and h.*

Proof: (1) Follows from the definition of the cross-ratio. (2) Using (1), for $\eta \in \partial X$ such that $g^n(\eta)\to g^+$, and x on an invariant axis of g

$$[g^-, g^+, \eta, g(\eta)] = \Pi_{n\in\mathbb{Z}}[g^n(x), g^n(gx), \eta, g(\eta)]$$

$$= \Pi_{n\in\mathbb{Z}}[x, gx, g^{-n}(\eta), g^{-n}(g\eta)] = [x, gx, g^-, g^+] = \lambda(g)^{-2}.$$

(3) Just note that the repelling and attracting fixed points of $h^n g^n$ converge to g^- and h^+ respectively, and the repelling and attracting fixed points of $g^n h^n$ converge to h^- and g^+ respectively. Then use Lemma 1. See [10] for rank one case. ∎

So combining Lemma 1 and Lemma 2, we obtain the following. Partial results are proved in [11].

Corollary 1. *Let $\phi : \Gamma_1\to\Gamma_2$ be an isomorphism between Zariski dense subgroups in a semisimple Lie groups G. Then the followings are equivalent.*

1. *Γ_1 and Γ_2 are conjugate.*
2. *Γ_1 and Γ_2 have the same marked length spectrum.*
3. *$\nu(g, h) = \nu(\phi(g), \phi(h))$ for any hyperbolic elements $g, h \in \Gamma_1$.*

As an application of this theorem, we answer in the affirmative a Margulis's question raised during the rigidity conference at Paris in June 1998. Denote $\lambda(\gamma)$ a unique element in a Weyl chamber A^+ which is conjugate to the hyperbolic component of γ in the Jordan decomposition, and denote $\log : A^+\to\mathfrak{a}^+$ a natural map from a Lie group to its Lie algebra.

Theorem 2. *Let G be a higher rank semisimple Lie group of noncompact type, and Γ_1, Γ_2 Zariski dense subgroups with a surjective homomorphism ϕ from Γ_1 to Γ_2. If $\log \lambda(\gamma) = k(\gamma) \log \lambda(\phi(\gamma))$ for all elements $\gamma \in \Gamma_1$, then Γ_1 and Γ_2 are conjugate.*

The followings are proved in [11]

Theorem 3. *Let $X_1 = H_1 \times H_2, X_2 = H_1' \times H_2'$ be rank two symmetric spaces of the product of rank one spaces. Suppose $\Gamma_i \subset Iso(X_i) = G_i$ are Zariski dense subgroups. Then the followings are equivalent.*

1. *There is a time-preserving conjugacy between $\Omega(\Gamma_1)$ and $\Omega(\Gamma_2)$.*
2. *Γ_1, Γ_2 have the same marked length spectrum.*
3. *The Mostow map Φ conjugating the actions between Λ_{Γ_1} and Λ_{Γ_2} preserves cross-ratio.*
4. *Γ_1 and Γ_2 are conjugate.*

Theorem 4. *Let M be a convex cocompact manifold with a metric g_1 which is a quotient of a rank one symmetric space of noncompact type. Let g_2 be another rank one symmetric metric which makes M convex cocompact. Then $I_{\mu_{BM}}(g_1, g_2) \geq \frac{\delta(g_1)}{\delta(g_2)}$ where $I_{\mu_{BM}}(g_1, g_2)$ is the geodesic stretch of g_2 relative to g_1 and the Bowen-Margulis measure μ_{BM} of g_1. Furthermore the followings are equivalent.*

1. *$I_{\mu_{BM}}(g_1, g_2) = \frac{\delta(g_1)}{\delta(g_2)} = 1$.*
2. *There is a time preserving conjugacy between $\Omega(g_1)$ and $\Omega(g_2)$.*
3. *Two manifolds have the same marked length spectrum.*
4. *g_1 and g_2 are isometric.*
5. *$\delta(g_1) = \delta(g_2)$ and the Thurston distance between g_1 and g_2 are zero.*

4. Applications

Theorem 5. *Let M and N be compact, strictly convex real projective manifolds with Hilbert metrics. If they have the same marked length spectrum then they are projectively equivalent.*

We prove an analogue of a Hitchin's conjecture [8] in dimension 3 [9]. The Teichmüller component is the component of a variety $\mathfrak{R}(\pi_1(M), PGL(n, \mathbb{R}))$ which contains the hyperbolic holonomy representations where the representation variety is $Hom(\pi_1(M), PGL(n, \mathbb{R}))/PGL(n, \mathbb{R})$. Denote $\mathfrak{B}(M)$ the set of strictly convex real projective structures on the manifold M and $\mathfrak{B}^0(M)$ a component of $\mathfrak{B}(M)$ containing hyperbolic structures.

Theorem 6. *The holonomy map $h : \mathfrak{B}^0(M) \rightarrow \mathfrak{R}(\pi_1(M), PGL(4, \mathbb{R}))$ is a homeomorphism onto the Teichmüller component if M is a hyperbolic 3-manifold. Here h is a map associating each convex real projective structure to its holonomy representation.*

Using above theorems we give a corollary.

Corollary 2. *Let $M = C_1/\Gamma_1$ and $N = C_2/\Gamma_2$ be compact strictly convex real projective n-manifolds. Then there exists a cross-ratio preserving equivariant homeomorphism between ∂C_1 and ∂C_2 iff M and N are projectively equivalent.*

Such type of a theorem is known between a negatively curved locally symmetric manifold and a quotient of a CAT(-1) space [2, 11].

The space $\mathbb{R}^{n+1,n}$is a vector space $\mathbb{R}^{2n+1}$ with a bilinear form $\mathbb{B}$ defined by

$$\mathbb{B}(X, Y) = x_1 y_1 + \cdots + x_{n+1} y_{n+1} - x_{n+2} y_{n+2} - \cdots - x_{2n+1} y_{2n+1}.$$

The group of isometries of this space is $O(n+1,n) \ltimes \mathbb{R}^{2n+1}$. For $(A,b) \in O(n+1,n) \ltimes \mathbb{R}^{2n+1}$, A is called a linear part, b a translational part. The nullcone

$$\mathcal{N} = \{x \in \mathbb{R}^{2n+1} | \mathbb{B}(x,x) = 0\}$$

is invariant by $O(n+1,n)$. The orientation preserving determinant 1 subgroup $SO^0(n+1,n)$ of $O(n+1,n)$ denotes the identity component of $O(n+1,n)$.

An element $g \in SO(n+1,n)$ is called (purely) **hyperbolic** if g has real eigenvalues, counting multiplicities,

$$|\lambda_{-n}(g)| \le \cdots \le |\lambda_{-1}(g)| < \lambda_0(g) = 1 < |\lambda_1(g)| \le \cdots |\lambda_n(g)|$$

such that $\lambda_i(g)\lambda_{-i}(g) = 1$. Denote $\chi_i(g)$ the corresponding eigenvectors so that

1. $B(\chi_i(g), \chi_j(g)) = \delta_{i,-j}$,
2. for $i = 0$, $B(\chi_0(g), \chi_0(g)) = 1$ and $(\chi_0(g), \chi_{-n}(g), \cdots, \chi_{-1}(g), \chi_1(g), \cdots, \chi_n(g))$ is positively oriented.

We say (A,b) is hyperbolic if A is hyperbolic. For $h = (g,v) \in SO(n+1,n) \ltimes \mathbb{R}^{2n+1}$ hyperbolic, the **Margulis invariant** $\alpha(h)$ of h is

$$\mathbb{B}(v, \chi_0(g)).$$

If n is odd, $\alpha(h)\alpha(h^{-1}) > 0$.

In this paper we consider the marked Margulis invariant of a Zariski dense subgroup in $SO(n+1,n) \ltimes \mathbb{R}^{2n+1}$. Then we have

Theorem 7. *Let Γ_1 and Γ_2 be Zariski dense subgroups of $SO(n+1,n) \ltimes \mathbb{R}^{2n+1}$. Suppose $\phi : \Gamma_1 \to \Gamma_2$ is an isomorphism. If ϕ preserves Margulis invariant, then ϕ is a conjugation by a translation.*

When $n = 1$, the theorem is independently proved by [4]. An **affine deformation** of $\Gamma \subset SO(n+1,n)$ is a homomorphism $\phi : \Gamma \to SO(n+1,n) \ltimes \mathbb{R}^{2n+1}$ such that $\phi(A,b) = (A, u_\phi(A))$. An affine deformation is **proper** if its action on $\mathbb{R}^{2n+1}$ is proper. Also the purpose of the choice $SO(2,1) \ltimes \mathbb{R}^4$ is to extend the affine deformation action of $SO(2,1)$ on $\mathbb{R}^3$ to $\mathbb{R}^4$. Here we identify $SO(2,1)$ with

$$\begin{bmatrix} 1 & 0 \\ 0 & SO(2,1) \end{bmatrix}.$$

We want to investigate the analogue of quasifuchsian deformations of Fuchsian groups in a hyperbolic 3-space.

If $\phi : \Gamma \to SO(2,1) \ltimes \mathbb{R}^4$ is an affine deformation with $u = u_\phi \in Z^1(\Gamma, \mathbb{R}^4) \subset Z^1(\Gamma, \mathfrak{so}(3,1))$ such that

$$-\frac{tr(u(g)g)}{\sqrt{(trg-2)^2 - 4}} > 0$$

for all $g \in \Gamma$ hyperbolic, we call u_ϕ **positive**. Then we have

Theorem 8. *Suppose $\Gamma \subset SO(2,1)$ is a cocompact lattice. Then any affine deformation $\phi : \Gamma \to SO(2,1) \ltimes \mathbb{R}^4$ with u_ϕ positive is not proper.*

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