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QUASI-HOMOGENEOUS DOMAINS AND PROJECTIVE MANIFOLDS

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Abstract. To understand compact projectively flat manifolds we study quasi-homogeneous projective domains, especially strictly convex domains. We show that any strictly convex projective domain is an ellipsoid if its boundary is twice differentiable and every strictly convex quasi-homogeneous affine domain is affinely equivalent to a paraboloid. As a result for non convex domains, we show that if a quasi-homogeneous domain has a strictly convex point, then it is actually a strictly convex domain. Concerning the homogeneity of quasi-homogeneous domains, we show that if the boundary of a quasi-homogeneous convex projective domain is everywhere twice differentiable except possibly at finite points, then it is homogeneous. We also find out all possible types of shapes for developing image of a compact convex affine manifold with dim≤ 4 and from this we get some results about Markus conjecture.

A quasi-homogeneous domain is an open subset Ω of $\mathbb{R}P^n$ which has a compact subset $K \subset \Omega$ and a subgroup G of $Aut(\Omega)$ such that $GK = \Omega$, where $Aut(\Omega)$ is a subgroup of $PGL(n+1, \mathbb{R})$ consisting of all projective transformations preserving Ω. In particular, if a quasi-homogeneous domain Ω is a subset of $\mathbb{R}^n$ and G is a subgroup of $\mathrm{Aff}(n, \mathbb{R})$, we will call Ω a quasi-homogeneous affine domain. Note that if Ω is an affine domain of $\mathbb{R}^n$, then we can consider Ω as a projective domain via the well-known equivariant imbedding of $(\mathbb{R}^n, \mathrm{Aff}(n, \mathbb{R}))$ into $(\mathbb{R}P^n, PGL(n+1, \mathbb{R}))$. Certain domain in $\mathbb{R}^n$ is not quasi-homogeneous as an affine domain but quasi-homogeneous as a projective domain. So we will use the terminology 'quasi-homogeneous affine domain' and 'quasi-homogeneous projective domain' to avoid ambiguity.

Quasi-homogeneous projective(affine, resp.) domains are closely related to compact projectively(affinely, resp.) flat manifolds. A projectively(affinly, resp.) flat manifold M is a manifold which is locally modelled on the projective(affine, resp.) space with its natural projective(affine) geometry, i.e., M admits a cover of coordinate charts into the projective(affine, resp.) space $\mathbb{R}P^n(\mathbb{R}^n$, resp.) whose coordinate transitions are projective(affine, resp.) transformations. By an analytic continuation of coordinate maps from its universal covering $\tilde{M}$, we obtain a developing map from $\tilde{M}$ into $\mathbb{R}P^n(\mathbb{R}^n$, resp.) and this map is rigid in the sense that it is determined only by a local data. Therefore the deck transformation action on $\tilde{M}$ induces the holonomy action via the developing map by rigidity. As an affine domain is considered as a projective domain, an affinely flat manifolds also can be viewed as a

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projectively flat manifolds. Every hyperbolic manifold admits a canonical projective structure via the Klein model and it belongs to the class of convex projective manifolds which are quotients of convex domains in an affine space of $\mathbb{R}P^n$ by a properly discontinuous and free action of a discrete subgroup of $PGL(n+1, \mathbb{R})$.

From the definition of projectively flat manifolds we can see immediately that the developing image of a compact projectively(or affinely) flat manifold is a quasi-homogeneous projective(or affine) domain. Particularly when the developing map is a diffeomorphism onto the developing image Ω as in the compact convex projective manifold case, Ω is a dividable domain, where a dividable domain is a domain in $\mathbb{R}P^n$ whose automorphism group contains a cocompact discrete subgroup acting properly and freely. So the study about quasi-homogeneous projective(affine, resp.) domains may be helpful to understand projectively(affinely, resp.) flat manifolds. We will use throught this paper the technique developed by Benzécri in two dimensional case in [1].

As another special case of quasi-homogeneous convex affine domains, homogeneous convex affine domains have been classified completely by Vingberg in [32] as follows.

> *The set of all homogeneous properly convex affine domains and the set of all clans are in one-to-one correspondence (up to isomorphism).*

In the above statement a *properly convex* affine domain means a convex affine domain which does not contain any complete line and a *clan* means a finite dimensional algebra $\mathcal{L}$ over the real number field satisfying the following conditions:

1. $a(bc) - (ab)c = b(ac) - (ba)c$ for all $a, b, c \in \mathcal{L}$;
2. There exists a linear form α on $\mathcal{L}$ such that $\alpha(ab) = \alpha(ba)$ for all $a \neq 0 \in \mathcal{L}$;
3. For every $a \in \mathcal{L}$, the eigenvalues of the left multiplication $x \in \mathcal{L} \to ax \in \mathcal{L}$ are real.

Vinberg also showed with Kats the following in [33].

> *(1) There is a quasi-homogeneous properly convex domain which is not homogeneous.*
>
> *(2) If Ω is a properly convex quasi-homogeneous domain in $\mathbb{R}P^2$ such that the boundary of Ω is everywhere twice differentiable except possibly at a finite number of points, then Ω must be homogeneous, that is, Ω is either an ellipse or a triangle.*

We can prove the following theorem generalizing (2) of the above theorem using the scaling technique developed by Benzécri in [1].

Theorem 1. *Let Ω be a convex quasi-homogeneous projective domain such that the boundary of Ω is everywhere twice differentiable except possibly at a finite number of points. Then Ω is homogeneous. Furthermore Ω is projectively equvalent to one of the following:*

(i) $\mathbb{R}^+$
(ii) $\mathbb{R} \times \mathbb{R}^+$
(iii) *a triangle*
(iv) *a ball*
(v) $\mathbb{R}\times$ *a ball*
(vi) *an affine space*

One might like to think that the twice differentiability on some open subset of $\partial\Omega$ implies the homogeneity of Ω. This is true for strictly convex quasi-homogeneous domains, which can be proved later using theorem 3. But every strictly convex cone is a convex quasi-homogeneous domain which is twice differentiable on some neighborhood of a point in their maximal dimensional proper face and it is not homogeneous if it is not an elliptic cone. So a natural question can be formulated as follows: *Is Ω homogeneous if the boundary of a quasi-homogeneous convex domain Ω is twice differentiable on a dense open subset?* We think that it may be true and we can prove this so far only when Ω is a quasi-homogeneous affine domain with dimension ≤ 3.

For strictly convex quasi-homogeneous projective domains Kuiper [20] proved the following theorem.

> *If a strictly convex projective domain $\Omega \in \mathbb{R}P^2$ covers a compact projectively flat manifold then Ω is an ellipse or the boundary fails to be C^2 on a dense subset.*

Since every 2-dimensional properly convex quasi-homogeneous projective domain is either a triangle or a strictly convex domain, the properly convex quasi-homogeneous domains constructed by Kats and Vinberg which are not homogeneous must be strictly convex and so their boundary are C^1 and but not C^2 by the above theorem of Kuiper. In fact, these strictly convex quasi-homogeneous projective domains are all dividable, that is, they cover a compact convex projectively flat manifold. Goldman showed in [8] that any strictly convex projective structure on a surface arised from a hyperbolic structure on it.

We can generalize the above result of Kuiper to arbitrary dimensions showing the following theorem about strictly convex quasi-homogeneous domains.

Theorem 2. *Let Ω be a strictly convex quasi-homogeneous projective domain. Then*

1. *$\partial\Omega$ is at least C^1.*
2. *Ω is an ellipsoid if $\partial\Omega$ is twice differentiabe.*
3. *If $\partial\Omega$ is C^α on an open subset of $\partial\Omega$, then $\partial\Omega$ is C^α everywhere.*

We can prove this theorem using Benzécri's results in [1] and the convex function theory developed in higher dimension (See [25, 26, 27]).

Recently, Benoist [2] has announced independently that if a strictly convex domain Ω covers a compact projectively flat manifold, then $\partial\Omega$ is at least C^1 and furthermore Ω is an ellipsoid provided that $\partial\Omega$ is C^2. So theorem 3 can be considered as a generalization of this to quasi-homogeneous case.

As an another earlier result related to this problem, B. Colbois and P. Verovic proved in [5] that if Ω is a strictly convex (in the sense that Hessian is positive definite) domain with C^3 boundary which has a cofinite volume discrete subgroup action, then Ω is an ellipsoid.

Now we consider the quasi-homogeneous domains which may not be properly convex, that is, they may either be not convex or contain complete lines. As in (3) of Theorem 2, we can show that some local properties which are satisfied on a neighborhood of a subset of $\partial\Omega$ are actually satisfied globally even if a quasi-homogeneous domain Ω is not strictly convex. For the first step, we get the following.

Proposition 3. *Let Ω be a quasi-homogeneous domain in $\mathbb{R}P^n$ (respectively, $\mathbb{R}^n$). If Ω is strictly convex at some point $p \in \partial\Omega$, that is, there exists an open neighborhood U of p such that $\Omega \cap U$ is convex and $\partial\Omega \cap U$ has no line segment, then Ω is a strictly convex domain in $\mathbb{R}P^n$ (respectively, $\mathbb{R}^n$).*

Obviously all property which is satisfied on $\Omega \cap U$ for some neighborhood U of a boundary point of a quasi-homogeneous domain Ω is not satisfied on Ω. But we can show that many local properties imply the same global properties if U satisfies some conditions. In fact, if $\Omega \cap U$ is a *pocket*, which is the terminology introduced by Benzécri in [1], then many local properties which are satisfied on a pocket imply the same global properties (See [13]).

If we restrict our consideration to affine domains the situation becomes different more than somewhat. First we can show that there is only one strictly convex quasi-homogeneous affine domain in contrast with the fact that there are infinitely many strictly convex quasi-homogeneous projective domains.

Theorem 4. *Every strictly convex quasi-homogeneous affine domain in $\mathbb{R}^n$ is affinely equivalent to a paraboloid.*

That is, in affine case there is no strictly convex quasi-homogeneous affine domain which is not homogeneous. The following result of Vey [29] about properly convex affine dividable domains is also contrasted to the projective case.

> *Let Ω be a properly convex affine domain in $\mathbb{R}^n$. If a group $G < Aff(n, \mathbb{R})$ divide Ω, then Ω is a cone.*

This is a generalization of the following result due to Koszul.

> *Let Ω be a properly convex affine domain in $\mathbb{R}^n$. If a unimodular Lie group $G < Aff(n, \mathbb{R})$ acts on Ω transitively, then Ω is a cone.*

For another results for dividable convex affine domains, Benoist announced in [2] that any dividable properly convex cone $\Omega \subset \mathbb{R}^n$ with a discrete subgroup Γ of $Aff(n, \mathbb{R})$ which divide Ω is one of the following three cases:

(1) Ω *is a product.*

(2) Ω *is homogeneous.*

(3) Γ *is Zariski dense in* $GL(n, \mathbb{R})$.

Note that the homogeneous cone appeared in (2) must be symmetric by the results of Koszul [19].

To our best knowledge, there is no known example for irreducible quasi-homogeneous affine domain which is neither homogeneous nor dividable. So we may ask whether every irreducible quasi-homogeneous affine domain is either homogeneous or dividable.

Now let's state our results about convex affine dividable domains.

Theorem 5. *Let Ω be an affine domain in $\mathbb{R}^3$ which covers a compact convex affine manifold. Then Ω is affinely equivalent to one of the following :*

1. $\mathbb{R}^3$ *; complete case,*
2. $\mathbb{R}^2 \times \mathbb{R}^+$*,*
3. $\mathbb{R} \times$ *parabola,*
4. $\mathbb{R} \times$ *quadrant,*
5. *dividable properly convex cones of dimension 3; radiant case.*

This is already proved using another technique by Cho [4] and from this she classified compact convex affine manifolds modifying the crystallographic hull argument which was used by Fried and Goldman to classify compact complete affine manifolds in [7]. For dimension 4, we get the following.

Theorem 6. *Let* Ω *be an affine domain in* $\mathbb{R}^4$ *which covers a compact convex affine manifold. Then* Ω *is affinely equivalent to one of the following :*

1. $\mathbb{R}^4$ *; complete case,*
2. $\mathbb{R}^3 \times \mathbb{R}^+$*,*
3. $\mathbb{R}^2 \times$*parabola,*
4. $\mathbb{R}^2 \times$*quadrant,*
5. $\mathbb{R} \times$*paraboloid,*
6. $\mathbb{R} \times \mathbb{R}^+ \times$*parabola,*
7. $\mathbb{R} \times$*dividable properly convex cones of dimension* 3*,*
8. *dividable properly convex cones of dimension* 4 *; radiant case.*

These theorems are obtained by finding out all possible quasi-homogeneous convex affine domains with dimension ≤ 3. Observing these shapes for developing images of compact convex affine manifolds we can prove the corollary below related to the Markus conjecture that a compact affine manifold is complete if and only if it has parallel volume(See [21] or [3]). In fact, from theorem 5 and 6, we can see that the holonomy group of compact convex affine manifold with dimension ≤ 4 has a proper invariant algebraic set if its developing map is not onto. This observation implies the corollary if we use the work of Fried, Goldman and Hirsch(See [7] and [12]).

Corollary 7. *Let* M *be a compact convex affine manifold with dimension* n*. Then*

1. *For* $n = 3$*,* M *is complete if and only if it has parallel volume.*
2. *For* $n = 4$*,* M *is complete if it has parallel volume.*

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