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MASLOV INDICES IN PRACTICE

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ABSTRACT. We review the definitions for a few types of Maslov indices with their respective geometric implications. Specific calculations of them are provided for a Lagrangian immersion of the Klein bottle into $\mathbb{C}^2$.

0. Introduction

This note as a whole is written in the hope that it might be a help to those who just started learning symplectic geometry. In particular the example is rare in which explicit calculations of Maslov indices are done for a concrete example of Lagrangian immersion. We provide one in the last section, which is the main motivation of the note.

In the first section we consider the space $\mathcal{L}(n)$ of Lagrangian subspaces and introduces an isomorphism $\mu : \pi_1(\mathcal{L}(n)) \to \mathbb{Z}$ called the Maslov index. The section follows mostly the notations and the treatments of the subject of [M], §2.3 by D. McDuff and D. Salamon. However our proof that μ is an isomorphism is somewhat different from theirs in that we exploit the well known fiber bundle $O(n) \to U(n) \to \mathcal{L}(n)$ more intensively on the process. The second section provides definitions for three types of Maslov indices for a Lagrangian immersion $f : M \to N$, that is, the two homomorphisms $\mu : \pi_1(M) \to \mathbb{Z}$, $\mu : \pi_2(f) \to \mathbb{Z}$ and the Maslov-Viterbo index. Subsequently in the third section we show their role in a few remarkable statements of symplectic geometry. In the last section we consider a Lagrangian immersion of the Klein bottle and calculate the Maslov indices for this specific example.

1. The space $\mathcal{L}(n)$ of Lagrangian subspaces

Consider $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ with the usual real vector space structure. Let $e_1, e_2, \cdots, e_n$ denote the standard basis for $\mathbb{R}^n \times \{0\}$ and $f_1, f_2, \cdots, f_n$, that for $\{0\} \times \mathbb{R}^n$. Then let J_0 be the almost complex structure given by

$$J_0(e_i) = f_i \text{ and } J_0(f_i) = -e_i \text{ for } i = 1, 2, \cdots, n.$$

Note we have indeed that $J_0^2 = -I$ where I denotes the identity operator. We will identify linear operators on $\mathbb{R}^{2n}$ with the $2n \times 2n$ matrices keeping the basis $e_1, e_2, \cdots, e_n, f_1, f_2, \cdots, f_n$ in mind. Then the symplectic form ω_0 on $\mathbb{R}^{2n}$ is defined by

$$\omega_0(\xi, \eta) = -\xi^t J_0 \eta$$

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where $\xi, \eta \in \mathbb{R}^{2n}$ are understood as column vectors.

A Lagrangian subspace Λ of $\mathbb{R}^{2n}$ is an n dimensional(maximal) subspace on which ω_0 is a zero form. For instance $\Lambda_0 = \mathbb{R}^n \times \{0\}$ is a Lagrangian subspace. It must be clear that for a symplectic linear map, that is, for an automorphism Φ of $\mathbb{R}^{2n}$ such that $\omega_0(\Phi\xi, \Phi\eta) = \omega_0(\xi, \eta)$ we have that $\Phi\Lambda$ is again a Lagrangian subspace.

On the other hand, by the specification of the almost complex structure J_0, $\mathbb{R}^{2n}$ is an n dimensional complex vector space and we may identify it with $\mathbb{C}^n$ using the complex basis $e_1, e_2, \cdots, e_n$. Let U be any complex linear operator on $\mathbb{C}^n$. We may write $U = A + iB$ where A, B are $n \times n$ real matrices. Then for any $v + iw \in \mathbb{R}^{2n}$, $v, w \in \mathbb{R}^n \times \{0\}$, we have that $(A + iB)(v + iw) = (Av - Bw) + i(Bv + Aw)$ and therefore the complex matrix U corresponds to the real $2n \times 2n$ matrix

$$U_{\mathbb{R}} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

It is easy to verify that $U_{\mathbb{R}} J_0 = J_0 U_{\mathbb{R}}$, $(U^*)_{\mathbb{R}} = (U_{\mathbb{R}})^t$ and, if U' is another complex matrix, $(UU')_{\mathbb{R}} = U_{\mathbb{R}} U'_{\mathbb{R}}$ and in particular that the identity complex matrix corresponds to the identity real matrix. Furthermore if U is unitary in the sense that $U^*U = I$ then we have that

$$-(U_{\mathbb{R}})^t J_0 U_{\mathbb{R}} = -(U_{\mathbb{R}})^t U_{\mathbb{R}} J_0 = -J_0,$$

which shows that $U_{\mathbb{R}}$ is symplectic.

Let Λ be any Lagrangian subspace of $\mathbb{R}^{2n}$ and let

$$\begin{pmatrix} A \\ B \end{pmatrix}$$

be a $2n \times n$ real matrix where A, B are real $n \times n$ matrices whose column vectors form an orthonormal basis of Λ. Then that Λ is Lagrangian means $A^tB - B^tA = 0$ and that the columns are orthonormal means $A^tA + B^tB = I$. This means that $A + iB$ is unitary and we have that $(A + iB)_{\mathbb{R}}\Lambda_0 = \Lambda$, which proves that the group $U(n)$ of all unitary matrices acts on $\mathcal{L}(n)$ transitively. The subgroup $O(n)$ of all orthogonal matrices is the isotropy subgroup of $U(n)$ for Λ_0. Thus we have a fibration

$$O(n) \to U(n) \to \mathcal{L}(n).$$

Now consider the exact sequence

$$\pi_1(O(n)) \to \pi_1(U(n)) \to \pi_1(\mathcal{L}(n)) \to \pi_0(O(n)) \to 0.$$

Assume $n \geq 1$. Note that $\pi_1(U(n)) \cong \mathbb{Z}$ and the isomorphism can be given by the map $\det : U(n) \to S^1$ by means of the usual identification $\pi_1(S^1) \cong \mathbb{Z}$. Since $\pi_1(O(n)) \cong \mathbb{Z}_2$, it follows that $\pi_1(O(n)) \to \pi_1(U(n))$ is a trivial homomorphism. Thus we have a short exact sequence

$$0 \to \mathbb{Z} \to \pi_1(\mathcal{L}(n)) \to \mathbb{Z}_2 \to 0.$$

Now consider the map $\rho : U(n) \to S^1$ defined by $\rho(U) = det(U)^2$. Then ρ induces a well-defined map $\bar{\rho}$ on $\mathcal{L}(n)$ via the projection $p : U(n) \to \mathcal{L}(n)$: Assume we have $U_1\Lambda_0 = U_2\Lambda_0$ and $U_1, U_2 \in U(n)$. Then we have $U_1^{-1}U_2 \in O(n)$ from which it follows that $\rho(U_1) = \rho(U_2)$. Now let I denote the closed unit interval and consider the path $\alpha : I \to U(n)$, $\alpha(t) = e^{i\pi t} \oplus 1 \oplus \cdots \oplus 1$, understanding the isomorphism $\mathbb{R}^{2n} \cong \mathbb{C}^n = \mathbb{C} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C}$. Then $\alpha(t)\Lambda_0$, $t \in I$, is a loop on $\mathcal{L}(n)$ whose class in $\pi_1(\mathcal{L}(n))$ we write as x and $\bar{\rho}(\alpha(t)\Lambda_0) = \rho(U(t))$, $t \in I$, is a loop on S^1 which represents the standard generator of $\pi_1(S^1) \cong \mathbb{Z}$. On the other hand, $\beta : I \to U(n)$, $\beta(t) = e^{2i\pi t} \oplus 1 \oplus \cdots \oplus 1$ is a loop on $U(n)$ which represents the generator of $\pi_1(U(n))$. We note that $\rho\beta$ represents $2x \in \pi_1(\mathcal{L}(n))$. Then we have

$$\langle 2x \rangle < \langle x \rangle < \pi_1(\mathcal{L}(n)), \quad \pi_1(\mathcal{L}(n))/\langle 2x \rangle \cong \mathbb{Z}_2.$$

We conclude that $(\pi_1(\mathcal{L}(n)) : \langle x \rangle) = 1$, that is, $\pi_1(\mathcal{L}(n)) = \langle x \rangle$. This also shows that that $\bar{\rho}_* : \pi_1(\mathcal{L}(n)) \to \pi_1(S^1)$ is an isomorphism.

Definition 1.1. The Maslov index μ on $\mathcal{L}(n)$ is the homomorphism

$$\nu\bar{\rho}_* : \pi_1(\mathcal{L}(n)) \to \mathbb{Z}$$

where $\nu : \pi_1(S^1) \to \mathbb{Z}$ is the isomorphism given by the generator of $\pi_1(S^1)$ represented by the loop $e^{2\pi i t}$, $t \in I$.

We observe for a later use in calculation the following facts, which are straightforward to verify from the definition of Maslov index.

Lemma 1.2. *(1) The class in $\pi_1(\mathcal{L}(1))$ represented by the loop $\Lambda(t) = e^{i\pi t}\mathbb{R} \subset \mathbb{C} = \mathbb{R}^2$, $t \in I$, has Maslov index 1*
(2) If $n = n_1 + n_2$, for any loops Λ_1, Λ_2 respectively on $\mathcal{L}(n_1)$, $\mathcal{L}(n_2)$ let $\Lambda_1 + \Lambda_2$ denote the obvious loop in $\mathcal{L}(n)$. Then we have that

$$\mu([\Lambda_1 + \Lambda_2]) = \mu([\Lambda_1]) + \mu([\Lambda_2]).$$

2. Lagrangian immersions

A manifold N^{2n} is referred to as a *symplectic manifold* if a symplectic differential form Ω, that is, a nowhere degenerate closed 2-form on N is specified. For instance $\mathbb{R}^{2n}$ is regarded as a symplectic manifold as follows: For each $i = 1, 2, \cdots, n$, let $x_i : \mathbb{R}^{2n} \to \mathbb{R}$ denote the projection to the i-th coordinate and $y_i : \mathbb{R}^{2n} \to \mathbb{R}$, that to the $(n+i)$-th coordinate. Then we consider $\mathbb{R}^{2n}$ with the symplectic differential form

$$\Omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \cdots + dx_n \wedge dy_n.$$

An immersion $f : M^n \to N$ is *Lagrangian* if $df\, T_xM$ is a Lagrangian subspace of the symplectic vector space $T_{f(x)}N, \Omega_{f(x)}$.

Note that the pull-back bundle f^*TN on M is a symplectic bundle whose symplectic structure is given by pulling back of Ω by means of the natural bundle map $f^*TN \to TN$. Assume there is a trivialization of f^*TN,

$$F : f^*TN \to \mathbb{R}^{2n},$$

as a symplectic bundle. Note that such a trivialization exists if N itself is parallelizable as a symplectic manifold as in the case when $N = \mathbb{R}^{2n}$. We may regard df as a map from TM into f^*TN. Then we have a map

$$G : M \to \mathcal{L}(n),\ G(x) = F(dfT_xM).$$

Definition 2.1 The Maslov index of the Lagrangian immersion $f : M \to N$ is a homomorphism $\mu : \pi_1(M) \to \mathbb{Z}$ which is given as the composite

$$\pi_1(M) \xrightarrow{G_*} \pi_1(\mathcal{L}(n)) \xrightarrow{\mu} \mathbb{Z}$$

where $\mu : \pi_1(\mathcal{L}(n)) \longrightarrow \mathbb{Z}$ is the Maslov index defined in the previous section.

It is straightforward to see that the Maslov index $\mu : \pi_1(M) \to \mathbb{Z}$ is an invariant of the homotopy class of the trivialization $F : f^*TN \to \mathbb{R}^{2n}$ and the homotopy class of the immersion $f : M^n \to N$ through Lagrangian immersions.

Note that in the above definition we demanded that there should be a trivialization $F : f^*TN \to \mathbb{R}^{2n}$ of the symplectic bundle f^*TN, which cannot be met for every Lagrangian immersion f. In general the Maslov index can be defined as a map from $\pi_2(f)$. Note that the elements of $\pi_2(f)$ are the homotopy class of pairs (α, ρ) consisting of a map $\alpha : D^2 \to N$ and another $\rho : S^1 \to M$ such that $\alpha|_{S^1} = f\rho$. Then df induces a map:

$$\overline{df} : \rho^*TM \to \alpha^*TN,\ \overline{df}(z, v_{\rho(z)}) = (z, dfv_{\rho(z)}).$$

Since D^2 is contractible, there is a trivialization $F : \alpha^*TN \to \mathbb{R}^{2n}$ of the symplectic bundle and it is unique up to homotopy. Then we may define a map

$$\rho_\alpha : S^1 \to \mathcal{L}(n),\ \rho_\alpha(z) = F(\overline{df}(\rho^*TM)_z) \in \mathcal{L}(n).$$

Definition 2.2 The Maslov index $\mu : \pi_2(f) \to \mathbb{Z}$ is given by defining $\mu([\alpha, \rho])$ as the Maslov index of the class represented by $\rho_\alpha : S^1 \to \mathcal{L}(n)$ for any class $[\alpha, \rho] \in \pi_2(f)$.

Now we consider another type of Maslov index for Lagrangian immersions called the Maslov-Viterbo index.

Let $f : M \to N$ be as in the above assuming further that f is completely regular in the sense that it is proper and has no triple points and self-transverse in the sense that $dfT_pM + dfT_qM = T_xN$ if $f(p) = f(q) = x,\ p \neq q$.

Let $x, y \in N$ be two double points of f such that $f^{-1}\{x\} = \{p_0, p_1\}$, $f^{-1}\{y\} = \{q_0, q_1\}$. Assume $\alpha : I^2 \to N$ is a continuous map such that $\alpha(\partial I^2) \subset f(M)$, $\alpha(\{0\} \times I) = \{x\}$, $\alpha(\{1\} \times I) = \{y\}$ and $\alpha|_{I \times \{i\}}$ lifts with respect f to a path on M, say, ρ_i from p_i to q_i for each $i = 0, 1$. Call such α a *Viterbo square.*

Since I^2 is contractible, there is a trivialization of α^*TN,

$$F : \alpha^*TN \to \mathbb{R}^{2n}$$

as a symplectic bundle. Also let J be an almost complex structure on N compatible with the symplectic structure Ω, in the sense that $\Omega(\cdot, J\cdot)$ is a Riemannian metric, such that $JdfT_{p_1}M = dfT_{p_0}M$ and $JdfT_{q_1}M = dfT_{q_0}M$. It is well known that for a symplectic vector space the space consisting of all complex structures compatible with the symplectic structure is contractible. Thus considering the obstruction theory such J always exists and is unique up to homotopy. Note that J preserves Lagrangian subspaces of T_zN for each $z \in N$. Now define a map $g : \partial I^2 \to \mathcal{L}(n)$ as follows:

$$g(u) = \begin{cases} F\{(u, v) | v \in dfT_{\rho_0(u)}M\} \text{ if } u \in I \times \{0\} \\ F\{(u, v) | v \in JdfT_{\rho_1(u)}M\} \text{ if } u \in I \times \{1\} \\ F\{(u, v) | v \in JdfT_{p_1}M\} \text{ if } u \in \{0\} \times I \\ F\{(u, v) | v \in JdfJT_{q_1}M\} \text{ if } u \in \{1\} \times I \end{cases}$$

It must be clear that given a Viterbo square α of f the homotopy class of g is invariant of the choices of the trivialization $F : \alpha^*TN \to \mathbb{R}^{2n}$ and the almost complex structure J compatible with Ω. Furthermore it is also straightforward to see that the homotopy class of g is invariant of the homotopy class of α through Viterbo squares. Consider an identification $\partial I^2 \cong S^1$ by a homomorphism respecting the counterclockwise orderings. Then g defines a map $\bar{g} : S^1 \to \mathcal{L}(n)$.

Definition 2.3. The Maslov-Viterbo index $\mu(\alpha)$ of a Viterbo square α of a completely regular Lagrangian immersion f is defined as the Maslov index of the class represented by $\bar{g} : S^1 \to \mathcal{L}(n)$.

3. Geometric implications

Let N^{2n}, Ω be any symplectic manifold and $i : M^n \to N$ be a Lagrangian submanifold. Then one of the most intensely studied subjects of symplectic geometry is when M can separated from itself by an ambient *Hamiltonian flow* $\Phi_t, t \in I$, of N, which is the integral curve of time dependent vector field $X_t, t \in I$, where X_t is such that

$$\Omega(X_t, Y) = df_t(Y) \text{ for any } Y \in TN$$

for some smooth 1-parameter family of smooth functions $f_t : N \to \mathbb{R}$, $t \in I$. An obstruction can be given in terms of the Floer cohomology $HF^*_{\mathbb{Z}_2}(M : N)$. In fact, if $HF^*_{\mathbb{Z}_2}(M : N) \neq 0$, then there is no Hamiltonian flow Φ_t, $t \in I$, such that $M \cap \Phi_1(M) = \emptyset$

A calculation of the Floer cohomology gives an interesting result assuming $i : M \to N$ is *monotone*, which means the following: Note that the Maslov index $\mu : \pi_2(i) \equiv \pi_2(N, M) \to \mathbb{Z}$ has been introduced in the previous section. We may define another map $\mu_\Omega : \pi_2(N, M) \to \mathbb{R}$ as follows: For any class $\xi \in \pi_2(N, M)$, represent ξ by a smooth $\alpha : D^2 \to N$ and set $\mu_\Omega(\xi) \equiv \int_{D^2} \alpha^*\Omega$. Then it is straightforward to see that μ_Ω is well defined using the facts any continuous map(between pairs) can be approximated by a smooth map, Ω is closed and M is a Lagrangian submanifolds of N and by applying the Stoke's theorem. It is not hard either to show that μ_Ω is a homomorphism. Now we say that $i : M \to N$ is monotone if there is a $\lambda \geq 0$ such that $\mu_\alpha(\xi) = \lambda\mu(\xi)$ for all $\xi \in \pi_2(N, M)$.

For any Lagrangian immersion $f : M \to N$, let Σ_f denote the minimum of all positive $\mu(\xi)$, $\xi \in \pi_2(f)$ if μ is not trivial and set $\Sigma_M = 0$ if μ is trivial.

Theorem[Y.G. Oh]. *Let N^{2n}, Ω be a symplectic manifold and $i : M^n \hookrightarrow N$ be a closed monotone Lagrangian submanifold of N. Then the equality*

$$HF^k_{\mathbb{Z}_2}(M : N) \cong H^k(M; \mathbb{Z}_2)$$

holds for $k = 0, 1, 2, \cdots, \Sigma_M - 1$ if $\Sigma_i \geq n + 2$ and for $k = 1, 2, \cdots, n - 1$ if $\Sigma_i = n + 1$.

In fact, the Floer cohomology is defined so that $HF^{k+\Sigma_M}_{\mathbb{Z}_2}(M : N) \cong HF^k_{\mathbb{Z}_2}(M : N)$ for all integer k([O], p.308).

Corollary 3.1. *Assume M is a closed manifold. If a Lagrangian immersion $f : M \to \mathbb{R}^{2n}$ is l-homotopic to a monotone Lagrangian embedding, then we have*

$$1 \leq \Sigma_f \leq n.$$

Proof. Since any Lagrangian submanifold $i : M \hookrightarrow \mathbb{R}^{2n}$ can be separated from itself by a Hamiltonian flow we must have $FH^*_{\mathbb{Z}_2}(M : \mathbb{R}^{2n}) = 0$. Also note that by a result of M. Gromov([Gr], $0.4.A_2$, p.313) and the assumption that f is l-homotopic to monotone Lagrangian embedding we have $\Sigma_i \geq 1$.

In fact the result of Gromov mentioned above states that for any Lagrangian submanifold $M \subset \mathbb{R}^{2n}$ the restriction to M of the Liouville class $\lambda = -y_1 dx_1 - y_2 dx_2 + \cdots - y_n dx_n$ represents a nontrivial class in $H^1(M; R)$. Note that $d\lambda = \Omega_0$ and therefore closed 1-form on M. It is straightforward to see that this is equivalent to saying that the homomorphism $\mu_{\Omega_0} : \pi_2(\mathbb{R}^{2n}, M) \to \mathbb{R}$ above is not trivial. It follows, in particular, that there is no Lagrangian embedding into $\mathbb{R}^{2n}$ of a simply connected closed n-manifold.

Let $f : M^n \to \mathbb{R}^{2n}$ be a Lagrangian immersion. Note that the symplectic vector bundle $T\mathbb{R}^{2n}$ is equipped with the "symplectic" frame

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}; \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \cdots, \frac{\partial}{\partial y_n}$$

which gives a trivialization $F : T\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ as a symplectic vector bundle. The trivialization is unique up to homotopy since $\mathbb{R}^{2n}$ is contractible. Therefore as shown in the previous section, there is a Maslov index $\mu : \pi_1(M) \to \mathbb{Z}$ which depends only on *l-homotopy class* of f, that is, homotopy class of f through Lagrangian immersions.

Also note that since $\mathbb{R}^{2n}$ is contractible the connecting homomorphism $\partial : \pi_2(f) \to \pi_1(M)$ is an isomorphism and it follows eventually that the two Maslov indices, one defined on $\pi_2(f)$ and the other, on $\pi_1(M)$, coincides up to this identification.

Now we will provide examples in which the Maslov-Viterbo index plays essential roles.

For any manifold M^n, the cotangent vector bundle T^*M, as a smooth manifold, has a natural symplectic structure Ω defined as follows: Let $U \subset M$ be an open set with coordinate functions $x_i : U \to \mathbb{R}$, $i = 1, 2, \cdots, n$. Then we may regard dx_i's as 1-forms on $p^{-1}U \subset T^*M$, identifying them with p^*dx_i's, where p is the projection map of the cotangent vector bundle. Every element of $p^{-1}U$ can be written uniquely by $\sum y_i dx_i$, that is, $x_i p, y_i : p^{-1}U \to \mathbb{R}$, $i = 1, 2, \cdots, n$, form a coordinate system. Then we define a $1 - form$ λ_{can} on T^*M by

$$\lambda_{can} = \sum_{i=1}^{n} y_i dx_i,$$

which is independent of the choice of the coordinate system and therefore indeed defines a global 1-form. Subsequently define the symplectic structure Ω by

$$\Omega = -d\lambda_{can} = dx_i \wedge dy_i.$$

Then the zero section form a Lagrangian submanifold $M_0 \subset T^*M$. In fact any graph in T^*M of a 1-form is Lagrangian if it is closed. Let $f : M \to \mathbb{R}$ denote a Morse function, that is, a smooth function with only non-degenerate critical points.

Then the graph $M_1 \subset T^*M$ of df is also a Lagrangian submanifold. That the critical points of f is non-degenerate is equivalent to that M_0, M_1 meets transversely. Denote the index of f at a critical point x as a Morse function by $\mathrm{index}_f(x)$

Lemma[C. Viterbo]. *Let df vanishes on $x_0, x_1 \in M$ and let $\alpha : I^2 \to T^*M$ be a Viterbo square such that $\alpha(\{i\} \times I) = x_i$, $\alpha(I \times \{i\}) \subset M_i$, $i = 0, 1$, and $p(\alpha(I^2)) \subset \alpha(I \times \{0\})$. Then for the Maslov Viterbo index we have:*

$$\mu(\alpha) = \mathrm{index}_f(x_0) - \mathrm{index}_f(x_1).$$

The following might be a more intuitive example for the Maslov index: Let l_0, l_1 be smoothly and properly embedded arcs on $\mathbb{R}^2$ meeting transversely at two points. Then $M_0 = l_0 \times \mathbb{R}^{n-1} \times \{0\}$, $M_1 = l_1 \times \{0\} \times \mathbb{R}^{n-1}$ are Lagrangian subspaces of $\mathbb{R}^2 \times \mathbb{R}^{2(n-1)} \equiv \mathbb{R}^{2n}$ meeting transversely at two points. Let $\alpha : I^2 \to \mathbb{R}^{2n}$ be any Viterbo square such that $\alpha(I \times \{0\}) \subset M_i$, $\alpha(I \times \{1\}) \subset M_{1-i}$ for some $i = 0, 1$ and $\alpha|_{\partial I^2}$ respects the 'counterclockwise' orientations assuming $\alpha(I^2) \subset \mathbb{R}^2 \times \{0\} \times \{0\}$ without loss of generality. Then it is not hard to see that for the Maslov-Viterbo index we have that $\mu(\alpha) = 1$.

More generally let M_0, M_1 be Lagrangian submanifolds of N meeting transversely at two points. Assume there is a symplectic open disk $D \subset N$ such that $D \cap M_0 = l_0$ and $D \cap M_1 = l_1$ are arcs which are properly embedded in D and meets at two points. Let B_ϵ^{n-1} denote the open ϵ-ball in $\mathbb{R}^{n-1}$ centered at the origin for any $\epsilon > 0$.

Lemma 3.2. *For some $\epsilon > 0$ there is a symplectomorphism $\psi : D \times B_\epsilon^{n-1} \times B_\epsilon^{n-1} \to N$ so that ψ is an inclusion on $D \times \{0\} \times \{0\}$ and $\psi(l_0 \times B_\epsilon^{n-1} \times \{0\}) \subset M_0$, $\psi(l_0 \times \{0\} \times B_\epsilon^{n-1}) \subset M_1$ if and only if $\mu(\alpha) = 1$ where $\alpha : I^2 \to N$ is a Viterbo square which maps into D and $\alpha|_{\partial I^2}$ respects the counterclockwise orientations as a map into D.*

The above observation is included in an unpublished paper([B]).

4. An example

Before considering a Lagrangian immersion of the Klein bottle, we would like to give first a brief history on the existence of its Lagrangian embedding: A connected closed 2-manifold can be realized as a Lagrangian submanifold in $\mathbb{R}^4$ only if its Euler number is a multiple of 4 according to M. Audin([A]). Indeed all such surfaces are found to admit a Lagrangian embedding into $\mathbb{R}^4$ except for the Klein bottle by A. B. Givental([Gi]). Recently S. Nemirovski([N]) wrote a paper asserting that the Klein bottle cannot admit one.

We regard the Klein bottle K as the quotient space $\mathbb{R}^2/\Gamma$ where Γ is the subgroup of $\mathrm{Diffeo}(\mathbb{R}^2)$ generated by $a, b : \mathbb{R}^2 \to \mathbb{R}^2$ which are defined by

$$a(x, y) = (x + 2\pi, -y),\ b(x, y) = (x, y + 2\pi).$$

Then consider the map $f : \mathbb{R}^2 \to \mathbb{C}^2$ defined by

$$f(x, y) = (1 + \sin^2 \frac{x}{2})(\cos y\ e_1(x) + \sqrt{2} \sin y\ e_2(\frac{x}{2}))$$

where $e_1(x) = (\cos x, \sin x; 0, 0)$, $e_2(x) = (0, 0; \cos x, \sin x)$ and the usual real vector space structure of $\mathbb{C}^2$ is understood. It is immediate to check $f(x + 2\pi, -y) = f(x, y)$, $f(x, y + 2\pi) = f(x, y)$ for any $(x, y) \in \mathbb{R}^2$. Thus f induces a well-defined map on K, which we denote again by f. Also it is straightforward to see that f is a Lagrangian immersion. Furthermore $f : K \to \mathbb{C}^2$ has exactly two transverse self-intersections:

$$f(\frac{\pi}{2}, 0) = f(\frac{3\pi}{2}, \pi) = \frac{3}{2}(0, 1; 0, 0),$$
$$f(\frac{\pi}{2}, \pi) = f(\frac{3\pi}{2}, 0) = \frac{3}{2}(0, -1; 0, 0).$$

The Maslov index. We show that the smallest positive Maslov index of $f : K \to M$ is 3. Note that $H^1(K; \mathbb{Z}) \cong \mathbb{Z}$ and any $\alpha \in H^1(K; \mathbb{Z})$ is determined by $\alpha(a) \in \mathbb{Z}$ where a is one of the generators of $\Gamma \cong \pi_1(K)$. Thus the minimal Maslov index of K is the absolute value of the Maslov index of the loop $\rho : I \to \mathcal{L}(2)$ defined by

$$\rho(t) = \langle \frac{\partial f}{\partial x}(2\pi t, 0), \frac{\partial f}{\partial y}(2\pi t, 0) \rangle.$$

We have that

$$\frac{\partial f}{\partial x}(2\pi t, 0) = \sin \pi t \cos \pi t e_1(2\pi t) + (1 + \sin^2 \pi t) e_1'(2\pi t)$$
$$\frac{\partial f}{\partial y}(2\pi t, 0) = \sqrt{2}(1 + \sin^2 \pi t) e_2'(\pi t).$$

Note that $\frac{\partial f}{\partial x}(2\pi t, 0) \in T(\mathbb{C} \times 0)$ and $\frac{\partial f}{\partial y}(2\pi t, 0) \in T(0 \times \mathbb{C})$. Now by applying Lemma 1.2 we conclude that the Maslov index ρ is the sum of the Maslov indexes of two loops in $\mathcal{L}(1)$ and from this it is straightforward to see that the Maslov index of ρ is 3.

Recall Corollary 3.1 which asserts that for any monotone Lagrangian submanifold $i : M \hookrightarrow \mathbb{C}^n$ the minimal Maslov number Σ_i satisfies $1 \le \Sigma_i \le n$. Since $H_1(K; \mathbb{Z})$ is of rank 1, any Lagrangian embedding of K is automatically monotone if one exists. Therefore $f : K \to \mathbb{C}^2$ cannot be l-homotopic to a Lagrangian embedding through Lagrangian immersions since $\Sigma_f = 3$.

The Maslov-Viterbo index. The two curves $f(x, 0) = (1 + \sin^2 \frac{x}{2}) e_1(x)$, $f(x, \pi) = -(1 + \sin^2 \frac{x}{2}) e_1(x)$ all lie on the symplectic plane $\mathbb{C} \times 0$ and the intersections $p = f(\frac{\pi}{2}, 0) = f(\frac{3\pi}{2}, \pi)$ and $q = f(\frac{3\pi}{2}, 0) = f(\frac{\pi}{2}, \pi)$ of the two curves are non other than the two self-intersections of f.

Write l_0 for the arc $\{f(x, 0) | -\frac{\pi}{2} \le x \le \frac{\pi}{2}\}$, l_1 for $\{f(x, \pi) | \frac{\pi}{2} \le x \le \frac{3\pi}{2}\}$ and l_2 for $\{f(x, \pi) | -\frac{\pi}{2} \le x \le \frac{\pi}{2}\}$.

First we calculate the Maslov-Viterbo index for the pair l_0, l_2. Since $\mathbb{C}^2$ is contractible, any two Viterbo squares $u : I^2 \to \mathbb{C}^2$ such that $u(I \times \{0\}) = l_0, u(I \times \{1\}) = l_2$ and $u(\{0\} \times I) = \{q\}, u(\{1\} \times I) = \{p\}$ are homotopic to each other through Viterbo squares. Note that such a Viterbo square respects the orientations. Also note that we may use the global natural trivialization of the symplectic bundle $T\mathbb{C}^2$ for the evaluation.

Note that $\frac{\partial f}{\partial x}(x, 0) = \sin \frac{x}{2} \cos \frac{x}{2} e_1(x) + (1 + \sin^2 \frac{x}{2}) e_1'(x) = -\frac{\partial f}{\partial x}(x, \pi)$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, lie in $T(\mathbb{C} \times 0)$ while $\frac{\partial f}{\partial y}(x, 0) = -\sqrt{2}(1 + \sin^2 \frac{x}{2}) e_2(\frac{x}{2}) = -\frac{\partial f}{\partial y}(x, \pi)$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, are in $T(0 \times \mathbb{C})$. Therefore the Maslov-Viterbo index is the sum of the Maslov indexes of the two loops in $\mathcal{L}(1)$ given as in the definition in Section 2 above. The one coming from $\frac{\partial f}{\partial x}(x,0)$, $\frac{\partial f}{\partial x}(x,\pi)$ has Maslov index 1 as can be seen as follows: It is the Maslov-Viterbo index for two 1-dimensional Lagrangian subspaces in $\mathbb{C}$ with exactly two transverse intersections and we may deform the subspaces by a symplectomorphism to Lagrangian subspaces so that one is a straightline segment and the other is a circle which meets it orthogonally. It is immediate to see that this has Maslov-Viterbo index 1. On the other hand, to calculate the Maslov index for the loop in $\mathcal{L}(1)$ coming from $\frac{\partial f}{\partial y}(x,0)$, $\frac{\partial f}{\partial y}(x,\pi)$, we note that at the intersections p, q, the two vectors $\frac{\partial f}{\partial y}(x,0)$, $\frac{\partial f}{\partial y}(x,\pi)$ are perpendicular and conclude that the loop given by $\langle e_2(\pi t)\rangle$, $t \in I$, is the one to be considered to calculate the portion of the Maslov index. Its Maslov index is clearly 1. Thus the Maslov-Viterbo index for the pair l_0, l_2 is 2. In view of Lemma 3.2, the fact that the pair does not have the Maslov- Viterbo index 1 reflects the fact the two intersections have the same intersection number when the orientation data are transfered by l_0, l_2, which means one cannot do the Whitney trick with l_0, l_2 and the disk bounded by the two on $\mathbb{C} \times 0$.

Now a similar calculation shows that the Maslov-Viterbo index for the pair l_0, l_1 is 1. This is also consistent with the fact that the two intersections have intersection numbers of opposite sign when the orientation data are transfered by l_0, l_1. In fact the Whitney trick with the triple l_0, l_1 and the region of $\mathbb{C} \times 0$ bounded by $l_0 \cup l_1$(or the triple of their slight extensions) can actually be performed to obtain a smooth embedding of the Klein bottle which is not necessarily Lagrangian .

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