THE SPACE OF FOURIER HYPERFUNCTIONS AS AN INDUCTIVE LIMIT OF HILBERT SPACES

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Abstract. We research properties of the space of measurable functions square integrable with weight \(\exp(2\nu |x|)\), and those of the space of Fourier hyperfunctions. Also we show that the several embedding theorems hold true, and that the Fourier-Laplace operator is an isomorphism of the space of strongly decreasing Fourier hyperfunctions onto the space of analytic functions extended to any strip in \(\mathbb{C}^n\) which are estimated with the aid of a special exponential function \(\exp(\mu |x|)\).

§0. Introduction

We introduced the following in [5]:

Let \(F_{(h,\nu)}\) be the space of continuously differentiable functions \(\varphi(x)\) for which the norm

\[|\varphi|_{(h,\nu)} = \sup_{x \in \mathbb{R}^n, \alpha} \frac{|\partial^\alpha \varphi(x)| \exp(\nu |x|)}{h^{-|\alpha|} \alpha!}, \quad h > 0, \nu \in \mathbb{R}\]

is finite. Then the spaces \(F_{(h,\nu)}\) the (continuous) embeddings

\[F_{(h,\nu)} \subset F_{(h',\nu')}, \quad h \geq h', \nu \geq \nu'\]

take place.
By virtue of (0.2), we can define the spaces \( G \) and \( M \) with the aid of the operations of projective and inductive limits:

\[
G = \bigcap_{h,\nu} F(h,\nu),
\]

\[
M = \bigcap_{h>0} F(h,\infty), \quad F(h,\infty) = \bigcup_{\nu} F(h,\nu).
\]

Let the space \( G' \) be a space of continuous linear functionals on \( G \). Since the embeddings (0.2) induce the adjoint embeddings

\[
(F(h',\nu'))' \subset (F(h,\nu))', \quad h \geq h' > 0, \quad \nu \geq \nu',
\]

The space \( G' \) regarded as a vector space coincides with the union of \( (F(h,\nu))' \):

\[
G' = \bigcup_{h,\nu} (F(h,\nu))'.
\]

The right-hand space can be equipped with the topology of inductive limit, and in the left-hand space we can introduce the topology of the strong conjugate space of \( G \).

Note that \( G' \) is reflexive and regular inductive limit, which implies the coincidence of the two above-mentioned topologies in \( G' \). The regularity of \( G' \) implies that for each bounded set \( B \subset G' \) there are real numbers \( h \) and \( \nu \) such that \( B \subset (F(h,\nu))' \).

In this paper, making use of the same method as in [3], we research the structure of an inductive limit of Hilbert spaces for the space of Fourier hyperfunctions introduced in [5].

In §2, we research properties of the reflexive Hilbert space \( H(\nu) \) of measurable functions square integrable with weight \( \exp(2\nu|x|) \) (Proposition 2.1). We show that the space \( G \) is dense in \( H = H_{<0>} \) (Theorem 2.3). Since the Fourier operator \( \mathcal{F} : G' \to G' \) is one-to-one and transforms the subset \( \mathcal{G} \subset \mathcal{G}' \) into itself and \( \mathcal{G} \subset H_{(<s>} \subset \mathcal{G}' \), we introduce the space \( H^{(s)} \), which is the image of \( H_{<s>} \) under the operator \( \mathcal{F}^{-1} \).

In §3, §4, we introduce the space \( H^{(s)}_{<\nu>} \), which consists of Fourier hyperfunctions in \( H^{(-\infty)} \) such that \( \delta_{s,N}(D)f \in H_{<\nu>} \), \( 0 \leq \nu < N \) for pseudodifferential operator \( \delta_{s,N}(D) \) with symbol \( \delta_{s,N}(\zeta) = \exp(s\sum_{k=1}^{n} ...} \)
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\[(N^2 + \zeta_k^2)^{1/2}, |Im \zeta_k| < N\]. And we research properties of Fourier hyperfunctions in this spaces (Proposition 3.1, 3.2 and 4.3). Also in Proposition 3.3, 4.1 and 4.2 we show the following embedding theorems: For \(h, \nu > 0\),

\[F_{(h, \pm \nu + \epsilon)} \subset H_{\pm \nu}^{(h/2)} \subset F_{(h/4, \pm \nu)}, \ \epsilon > 0.\]

Lastly, we introduce the spaces \(\mathcal{G}, \mathcal{O}, \mathcal{M}, \mathcal{O}'\) and \(\mathcal{G}'\) endowed with topologies of projective and injective limits of the Hibert spaces \(H_{\nu}^{(s)}\) and show that the Fourier-Laplace transform \(\mathcal{F} : \mathcal{O}' \to \mathcal{M}\) is an isomorphism of vector spaces (Proposition 4.4).

§1. Preliminaries

As a norm in \(\mathbb{R}_x^n\) (in \(\mathbb{C}_x^n\), resp.) we take \(|x| = \sum_{j=1}^n |x_j|\) (\(|z| = \sum_{j=1}^n |z_j|\), resp.), and the volume element \(dx = dx_1 \cdots dx_n\) is fixed. Put \(D = (D_1, \ldots, D_n); D_k = i^{-1} \partial_k, \ \partial_k = \partial / \partial x_k, \ k = 1, \ldots, n, \ i = \sqrt{-1}.\)

We denote by \(\mathbb{R}_x^n\) the dual space of \(\mathbb{R}_x^n\). Let \(\xi = (\xi_1, \ldots, \xi_n)\) be coordinates in \(\mathbb{R}_x^n\) such that the duality is expressed by the bilinear form \(<x, \xi> = x_1\xi_1 + \cdots + x_n\xi_n\). If \(\beta = (\beta_1, \ldots, \beta_n)\) and \(\gamma = (\gamma_1, \ldots, \gamma_n)\) are multi-indices of nonnegative integers, then \(|\beta| = \beta_1 + \cdots + \beta_n, \beta + \gamma = (\beta_1 + \gamma_1, \ldots, \beta_n + \gamma_n), \beta! = \beta_1! \cdots \beta_n!, \xi^\beta = \xi_1^{\beta_1} \cdots \xi_n^{\beta_n}, D^\beta = D_1^{\beta_1} \cdots D_n^{\beta_n}, \) and \(\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}\).

Let \(E_1\) and \(E_2\) be topological vector spaces embedded in a topological space \(E\). Denote by \(E_1 \cap E_2\) and \(E_1 + E_2\) the subspace of elements of \(E_1\) being contained in \(E_2\) and the space of sums \(\varphi_1 + \varphi_2, \ \varphi_1 \in E_1, \ \varphi_2 \in E_2\), respectively. The topologies of \(E_1\) and \(E_2\) induce topologies in \(E_1 \cap E_2\) and \(E_1 + E_2\). In case \(E_1\) and \(E_2\) are Banach spaces, the Banach norm

\[|\varphi, E_1 \cap E_2| = |\varphi, E_1| + |\varphi, E_2|\]

and the norm

\[|\varphi, E_1 + E_2| = \inf_{\varphi_1 + \varphi_2 = \varphi} (|\varphi_1, E_1| + |\varphi_2, E_2|)\]

are defined on \(E_1 \cap E_2\) and \(E_1 + E_2\), respectively.

Let \(I\) denote an open unit cube in \(\mathbb{R}_x^n\):

\[I = \{\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}_x^n \mid |\omega_j| < 1, j = 1, \ldots, n\}\]

and let \(I^{(\kappa)}, \ \kappa = 1, \ldots, 2^n\), be the vertices of the cube, i.e., the various vector whose coordinates assume the values \(\pm 1\).

We introduce some theorems and propositions to need in this paper which are founded in [5].
Theorem 1.1. $f(x) \in F(h,\nu)$ if and only if $f(x)$ can be continued holomorphically to the tube domain $D_h = \{ x + yi \in \mathbb{C}^n \mid |y_j| < h, j = 1, 2, \cdots, n \}$ such that

$$|f(x + yi)| \leq C \exp(-\nu|x|).$$

Let $\nu > 0$. Let $F(\nu,s)$ denote the Banach space of functions $\psi(\zeta)$ holomorphic in the tube domain $D_\nu$ and having a finite norm

$$|\psi|^{(\nu,s)} = \sup_{\zeta \in D_\nu} \exp(s|\zeta|) |\psi(\zeta)|.$$

Proposition 1.2. The map $F(h,\nu) \to F^{(h,\nu)} : f(x) \to f(x + yi)$ is a topological isomorphism and there are constants $C_1, C_2 > 0$ such that

$$C_1 |f|^{(h,\nu)} \leq |f|^{(h,\nu)} \leq C_2 |f|^{(h,\nu)}.$$

Remark. It follows from Proposition 1.2 that

$$(1.2) \quad \mathcal{M} = \bigcap_{h > 0} F^{(h,-\infty)}.$$

Proposition 1.3. For $h, \nu > 0$ we have

$$(1.3) \quad F(h,\nu) = \bigcap_{\kappa = 1}^{2^n} F_{[h,\nu I^{(\kappa)}]}$$

and (0.1) is equivalent to the natural norm of the right-hand space of (1.3):

$$(0.1') \quad \sum_{\kappa = 1}^{2^n} |\varphi|_{[h,\nu I^{(\kappa)}]}.$$
Theorem 1.5. The spaces $\mathcal{G}$ and $\mathcal{G}'$ are Fourier self-dual:

\begin{equation}
\mathcal{F}\mathcal{G} = \mathcal{G}, \mathcal{F}\mathcal{G}' = \mathcal{G}'.
\end{equation}

Theorem 1.6. If $\varphi \in F(h, \nu)$, $h, \nu > 0$, then for any $\zeta \in D_\nu$, the absolutely convergent Fourier-Laplace integral is defined:

$$
\mathcal{F} : \varphi(x) \rightarrow \hat{\varphi}(\zeta) = (2\pi)^{-n/2} \int \exp(-i < x, \zeta>) \varphi(x) dx,
$$

and Parseval’s inequalities hold:

\begin{equation}
C_1 |\hat{\varphi}|^{(\nu,h)} \leq |\varphi|(h, \nu) \leq C_2 |\hat{\varphi}|^{(\nu,h)}.
\end{equation}

Theorem 1.7. The Fourier-Laplace transform operator determines an isomorphism

$$
\mathcal{F}\mathcal{G} = \bigcap_{h, \nu > 0} F(\nu, h).
$$

Remark. Regarded all function in $\mathcal{M}$ ($\mathcal{G}$ resp.) as an entire function estimated with a special exponential function $\exp(\mu|x|)$ (with any exponential function $\exp(\mu|x|)$ resp.), $\mathcal{G}$ is an ideal in $\mathcal{M}$, i.e., each element in $\mathcal{M}$ is a multiplier on $\mathcal{G}$.

By virtue of (0.2), we can define the spaces $\mathcal{O}$ with the aid of the operations of projective and inductive limits:

\begin{equation}
\mathcal{O} = \bigcup_{\nu} F_{(\infty, \nu)}, \quad F_{(\infty, \nu)} = \bigcap_{h > 0} F_{(h, \nu)}
\end{equation}

The spaces $\mathcal{O}$ consists of analytic functions extended to $\mathbb{C}^n$ whose functions increase, for $|Rez| \rightarrow \infty$, not stronger than an exponential function $\exp(\mu|Rez|)$.

Remark. Since $\exp(i \sum_{k=1}^{n} x_k^2)$ belnogs to $\mathcal{M}$, but does not belong to $\mathcal{O}$, $\mathcal{O}$ is a proper subspace of $\mathcal{M}$.

If $f \in F(h, \nu)$, $g \in F(h, -\nu + \epsilon)$, $h, \epsilon > 0$, then the bilinear form

\begin{equation}
(f, g) = \int f(x)g(x)dx
\end{equation}
is defined and depends continuously on \( f \) and \( g \) (in the corresponding topologies). Hence,
\[
(1.8) \quad \mathcal{G} \subset F(h,\nu) \subset (F(h,-\nu+\epsilon))' \subset \mathcal{G}'
\]
and the embeddings
\[
(1.9) \quad \mathcal{G} \subset \mathcal{O} \subset \mathcal{G}'.
\]
take place.

Denote by \( \mathcal{O}' \) the space of continuous linear functionals on \( \mathcal{O} \). \( \mathcal{O}' \) regarded as a vector space can be identified with the projective limit of the conjugate spaces \( (F(\infty,\nu))' \). The latter, when treated as vector spaces, are identified with the inductive limit \( \bigcup_h (F(h,\nu))' \). Thus,
\[
(1.10) \quad \mathcal{O}' = \bigcap_{\nu} \bigcup_h (F(h,\nu))'.
\]

The space \( \mathcal{O}' \) are called the space of strongly decreasing Fourier hyperfunctions.

§2. The spaces \( H_{<\nu>} \)

Let \( H_{<\nu>} (\nu \in \mathbb{R}) \) denote the space of measurable functions square integrable with weight \( \exp(2\nu|x|) \). The corresponding norm is written
\[
(2.1) \quad \|f\|_{<\nu>} = ((f,f)_{<\nu>})^{1/2} = \|\exp(\nu|x|)f\|_{L^2}.
\]
Then we see that
\[
\mathcal{G} \subset F(h,\mu+\epsilon) \subset H_{<\mu>} \subset (F(h,-\mu+\epsilon))' \subset \mathcal{G}', \quad \epsilon > 0.
\]

REMARK. If \( \exp(\nu|x|) \) in (2.1) is replaced by \( \delta_{\nu,N}(x) = \exp(\nu \sum_{k=1}^n (N^2 + x_k^2)^2) \), we note that (2.1) is equivalent to the norm \( \|\delta_{\nu,N}(x)f\|_{L^2} \).

Let \( H_{<0>} = H \). Then the mapping \( f \rightarrow f \exp(\nu|x|) \) determines an isometric isomorphism of \( H_{<\nu>} \) and \( H \). It follows that \( H_{<-\nu>} \) and the Banach conjugate space of \( H_{<\nu>} \) are isometrically isomorphic, i.e.,
\[
(2.2) \quad (H_{<\nu>})' = H_{<-\nu>}
\]
Consequently, \( H_{<\nu>} \) is a reflexive Banach space.

We can define in \( H_{<\nu>} \) the scalar product
\[
(2.3) \quad (f,g)_{<\nu>} = \int \exp(2\nu|x|)f(x \overline{g(x)})dx
\]
to which the norm (2.1) corresponds. In other words, \( H_{<\nu>} \) is a reflexive Hilbert space.
Proposition 2.1. 
(i) For \( \nu > 0 \) we have

\[
H_{<\nu>} = \bigcap_{\kappa = 1}^{2^n} H_{[\nu I^I(\kappa)]}
\]

and (2.1) is equivalent to the natural norm

\[
\sum_{\kappa = 1}^{2^n} \| f \|_{[\nu I^I(\kappa)]}
\]

of the right-hand space of (2.4).

(i') For \( \nu > 0 \) the space \( H_{<\nu>} \) consists of those and only those elements of the intersection \( \bigcap_{\Gamma \in \nu I} H_{[\Gamma]} \) for which the norm

\[
\| f \|_{<\nu>} = \sup_{\Gamma \in \nu I} \| f \|_{[\Gamma]}
\]

is finite.

(ii) For \( \nu < 0 \) we have

\[
H_{<\nu>} = \bigoplus_{\kappa = 1}^{2^n} H_{[\nu I^I(\kappa)]},
\]

and (2.1) is equivalent to the norm

\[
\inf_{f=f_1+\cdots+f_n} \sum_{\kappa} \| f_\kappa \|_{[\nu I^I(\kappa)]}.
\]

Proof. (i) It is obvious that

\[
\exp(\nu|x_i|) \leq \exp(\nu x_i) + \exp(-\nu x_i) \leq 2 \exp(\nu|x_i|).
\]

Multiplying these inequalities for \( i = 1, \ldots, n \) we find

\[
\exp(\nu|x|) \leq \sum_{\kappa = 1}^{2^n} \exp(<\nu I^I(\kappa), x>) \leq 2^n \exp(\nu|x|),
\]

which implies that (2.1) and (2.5) are equivalent for \( \nu > 0 \).
Let $\Gamma = (\omega_1, \ldots, \omega_n)$ and let $|\omega_j| < \nu, j = 1, \ldots, n$. Multiplying the inequalities
\begin{equation}
\exp(\omega_i x_i) \leq \exp(\nu x_i) + \exp(-\nu x_i) \leq 2 \exp(\nu |x_i|)
\end{equation}

for $i = 1, \ldots, n$, we obtain
\begin{equation}
\exp(\langle \Gamma, x \rangle) \leq \sum_{\kappa=1}^{2^n} \exp(\langle \nu I^{(\kappa)}, x \rangle) \leq 2^n \exp(\nu |x|),
\end{equation}

whence
\begin{equation}
\left\| f \right\|_{\nu} \leq \sum_{\kappa=1}^{2^n} \left\| f \right\|_{\nu I^{(\kappa)}} \leq \left( \int \exp(2\rho |x|) |f|^2 dx \right)^{1/2} \leq \left\| f \right\|_{\nu}.
\end{equation}

Conversely, let $\left\| f \right\|_{\nu} < \infty$. Given $x \in \mathbb{R}^n$, we take $\Gamma = (\epsilon_1 \rho, \ldots, \epsilon \rho)$, $\rho < \nu$, in (2.10), where $\epsilon_i = \pm 1$ and the sign of $\epsilon_i$ coincides with that of $x_i$. Then we derive
\begin{equation}
\left( \int \exp(2\rho |x|) |f| dx \right)^{1/2} \leq \left\| f \right\|_{\nu}.
\end{equation}

for all $\rho < \nu$. By continuity, this inequality is retained for $\rho = \nu$ as well.

(ii) By virtue of the obvious inequality
\begin{equation}
\exp(\nu |x|) \leq \exp(\langle \Gamma, x \rangle), \quad \forall \nu < 0, \quad \Gamma = (\omega_1, \ldots, \omega_n), \quad |\omega_j| \leq |\nu|,
\end{equation}

the spaces $H_{\nu I^{(\kappa)}}, \kappa = 1, 2, \ldots, 2^n$, and, consequently, their linear hull as well are embedded in $H_{\nu}$. And if $\psi = \sum_{\kappa=1}^{2^n} \psi^{(\kappa)}$, $\psi^{(\kappa)} \in H_{\nu I^{(\kappa)}}$, then, by triangle inequality,
\begin{equation}
\left\| \psi \right\|_{\nu} \leq \sum_{\kappa=1}^{2^n} \left\| \psi^{(\kappa)} \right\|_{\nu I^{(\kappa)}}.
\end{equation}

Taking the infimum in the right-hand side over all the representation $\psi = \sum_{\kappa=1}^{2^n} \psi^{(\kappa)}$, we prove that the right-hand side space of (2.6) is embedded into the left-hand side space.

To prove the opposite embedding we construct a system of functions $\chi^{(\kappa)}, \kappa = 1, \ldots, 2^n, \chi^{(\kappa)} \geq 0$, possessing the following properties:

(a) $\sum_{\kappa=1}^{2^n} \chi^{(\kappa)}(x) = 1$

(b) If $\psi \in H_{\nu}$, then $\chi^{(\kappa)} \psi \in H_{\nu I^{(\kappa)}}$, and $\left\| \chi^{(\kappa)}(x) \psi \right\|_{\nu I^{(\kappa)}} \leq \text{const} \left\| \psi \right\|_{\nu}$.
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The embedding of the left-hand space of (2.6) into the right-hand side space is a trivial consequence of (a) and (b). If \( x_{i_1}, \ldots, x_{i_k} \geq 0 \) and \( x_{i_{k+1}}, \ldots, x_{i_n} < 0 \), let \((\epsilon_1, \ldots, \epsilon_n)\) be the coordinates of a vertex \( I^{(\kappa)} \), where \( \epsilon_{i_1} = \cdots = \epsilon_k = 1 \) and \( \epsilon_{i_{k+1}} = \cdots = \epsilon_n = -1 \). Then we put

\[
\chi^{(\kappa)}(x) = \prod_{l=1}^{k} \exp(-x_{i_l}^2) \prod_{l=k+1}^{n} (1 - \exp(-x_{i_l}^2)).
\]

It is obvious that (a) is fulfilled. Since

\[
< I^{(\kappa)}, x > = \sum_{l=1}^{k} x_{i_l} - \sum_{l=k+1}^{n} x_{i_l} = |x|,
\]

it is clear that (b) holds, whence the proposition is proved. \( \square \)

For \( \delta_{s,N}(\zeta) = \exp(s \sum_{k=1}^{n} (N^2 + \zeta_k^2)^{1/2}) \), \(|Im\zeta_k| < \nu < N\) we denote by \( H^{(s)}_{<\nu>} \) the space of functions \( \psi(\zeta) \) holomorphic in the tube domain \( D_\nu \) and possessing a finite norm

\[
\|\psi\|^{<\nu}_{(s)} = \sup_{|\omega_j| < \nu, j = 1, \ldots, n} \left( \int |\delta_{s,N}(\xi + i\omega)|^2 d\xi \right)^{1/2}.
\]

Note that the Fourier-Laplace operator is defined in the spaces \( H^{<\nu>} \) for any \( \zeta \in D_\nu \) if \( \nu > 0 \).

**THEOREM 2.2.** The Fourier-Laplace operator determines the isomorphism

\[
\mathcal{F} H^{<\nu>} = H^{<\nu>}
\]

under which the norm (2.1') goes into (2.12) (for \( s = 0 \)).

**THEOREM 2.3.** \( \mathcal{G} \) is dense in \( H \).

**PROOF.** First of all, we show that \( \mathcal{G} \) is dense in \( F(\infty, 0) \cap H \) relative to the norm \( \| \cdot \|_{L^2} \).

Let \( f \in F(\infty, 0) \cap H \). Then we see that \( f(x) \exp(-j^{-1} \sum_{k=1}^{n} x_k^2) \in \mathcal{G} \). It follows from the Lebesgue’s dominated convergence theorem that

\[
\lim_{j \to \infty} \| f(x) \exp(-j^{-1} \sum_{k=1}^{n} x_k^2) - f(x) \| = 0.
\]
Therefore it follows.

Next, we show that \( F(\infty, 0) \cap H \) is dense in \( H \).

Let \( f \in H \), and let \( \varphi_\epsilon(x) = \pi^{-n/2} \epsilon^{-n} \exp(-\sum_{k=1}^{n} (x_k / \epsilon)^2) \). Then we obtain from Young’s inequality that \( \| f * \varphi_\epsilon \| \leq \| f \| \). We can easily show from Theorem 1.1 that \( f_\epsilon(x) = f * \varphi_\epsilon(x) \) belongs to \( F(\infty, 0) \cap H \).

Since \( C_0^0 \) is dense in \( H \), for every \( \eta > 0 \) there exists a function \( \tilde{f} \in C_0^0 \) such that \( \| f - \tilde{f} \| < \eta / 2 \). Therefore we obtain

\[
\| f_\epsilon - f \| \leq \| f_\epsilon - \tilde{f}_\epsilon \| + \| \tilde{f}_\epsilon - \tilde{f} \| + \| \tilde{f} - f \|
\]

On the other hand, we have

\[
| \tilde{f}_\epsilon(x) - \tilde{f}(x) |
\]

\[
\leq \pi^{-n/2} \epsilon^{-n} \int | \tilde{f}(x-t) - \tilde{f}(x) | \exp(- \sum_{k=1}^{n} (t_k / \epsilon)^2) dt
\]

\[
= \pi^{-n/2} \epsilon^{-n} \int_{|t_k| \geq \sqrt{\epsilon}} | \tilde{f}(x-t) - \tilde{f}(x) | \exp(- \sum_{k=1}^{n} (t_k / \epsilon)^2) dt
\]

\[
+ \pi^{-n/2} \epsilon^{-n} \int_{|t_k| \leq \sqrt{\epsilon}} | \tilde{f}(x-t) - \tilde{f}(x) | \exp(- \sum_{k=1}^{n} (t_k / \epsilon)^2) dt
\]

\[
\leq 2\pi^{-n/2} \epsilon^{-n} \sup_x | \tilde{f}(x) | \int_{|t_k| \geq \sqrt{\epsilon}} \exp(- \sum_{k=1}^{n} (t_k / \epsilon)^2) dt
\]

\[
+ \sup_{|t_k| \leq \sqrt{\epsilon}} | \tilde{f}(x-t) - \tilde{f}(x) |.
\]

Since \( \tilde{f}(x) \) is uniformly continuous, \( \tilde{f}_\epsilon - \tilde{f} \rightarrow 0 \) uniformly, so \( \| \tilde{f}_\epsilon - \tilde{f} \| \rightarrow 0 \) as \( \epsilon \rightarrow 0_+ \). Therefore it follows, and hence proves the theorem.

**Remark.** The operator of multiplication by

\[
\exp(-\nu \sum_{k=1}^{n} (N^2 + x_k^2)^{1/2})
\]

generates an isomorphism of \( H \) and \( H_{<\nu>} \) under which the spaces \( G, F(\infty, 0) \cap H \) is preserved. Since \( G \) is dense in \( H \), it is dense in \( H_{<\nu>} \) as well, and \( H_{<\nu>} \) can be regarded as the completion of \( G \) with respect to \( \| \cdot \|_{<\nu>} \).
§3. The structure of a countably Hilbert space in $\mathcal{G}$

Note that $\mathcal{G} \subset H_{<s>} \subset \mathcal{G}'$. Then it follows from Theorem 1.5 that

$$\mathcal{G} \subset \mathcal{F}^{-1}H_{<s>} \subset \mathcal{G}'.$$ 

Let

$$H^{(s)} = \mathcal{F}^{-1}H_{<s>}.$$ 

Since the composition of $\mathcal{F}$ and $\mathcal{F}^{-1}$ is an identity operator in $\mathcal{G}'$, we have

$$\mathcal{F}H^{(s)} = H_{<s>}.$$ 

We introduce the norm

$$\|f\|^{(s)} = \|\mathcal{F}f\|_{<s>}$$ 

in the space $H^{(s)}$. Thus, $H^{(s)}$ consists of those functions $f \in \mathcal{G}'$, possessing Fourier transforms, which are square summable and whose norm $\|f\|^{(s)}$ is finite.

By pseudodifferential operators (PDO) in $\mathcal{G}$ are meant operators having the form

$$\begin{align*}
(a(D)\varphi)(x) &= (2\pi)^{-n/2} \int \exp(i < x, \xi >)a(\xi)\hat{\varphi}(\xi)d\xi. 
\end{align*}$$

Such an operator can be written as a composition of three operators:

$$\begin{align*}
(a(D)\varphi)(x) &= \mathcal{F}^{-1}_{\xi \rightarrow x}a(\xi)\mathcal{F}_{x \rightarrow \xi}\varphi. 
\end{align*}$$

The function $a(\xi)$ is called the symbol of the operator. We now consider the case $\varphi \in \mathcal{G}$ and take, as symbols $a(\xi)$, functions belonging to $\mathcal{M}$. Then, by virtue of Proposition 1.4 (ii) and Theorem 1.5, (3.2) is a composition of three continuous operators transforming $\mathcal{G}$ into itself and hence is a continuous operator from $\mathcal{G}$ into $\mathcal{G}$.

If $f \in \mathcal{G}'$ and $a(\xi) \in \mathcal{M}$, then we define by $a(D)f$ the functional

$$\begin{align*}
(a(D)f, \varphi) &= (f, a(-D)\varphi), \ \forall \varphi \in \mathcal{G}.
\end{align*}$$

Fix $\nu > 0$. Each function $a(\xi) \in F^{(\nu, -\infty)}$ is a multiplier on the space $F^{(\nu', \infty)}$, $\nu' < \nu$, and for $\varphi \in F_{(\nu, \nu')}$, we can define the PDO

$$\begin{align*}
a(D)\varphi &= (2\pi)^{-n/2} \int \exp(i < \xi + i\Gamma, x >)a(\xi + i\Gamma)\hat{\varphi}(\xi + i\Gamma)d\xi, \ \Gamma \in \nu'I.
\end{align*}$$
Then Theorem 1.1 and 1.6 imply that the operator
\[ a(D) : F_{(\infty, \nu')} \to F_{(\infty, \nu')}, \quad 0 < \nu' < \nu, \quad a(\zeta) \in F_{(\nu', -\infty)} \]
is continuous.

We now include the zeroth space \( H_{<\nu>} \), \( \nu > 0 \), in the scale \( \{H_{<\nu>^s}\} \) generated with respect to PDO. To this end we define the symbols

\[ (3.4) \quad \delta_{s,N}(\zeta) = \exp(s \sum_{k=1}^{n} (N^2 + \zeta_k^2)^{1/2}). \]

Note that if \( s \geq 0 \), then for \( |\text{Im}\zeta_k| < \nu < N \),

\[ (3.5) \quad C_1 \exp(s/\sqrt{2}|\zeta|) \leq |\delta_{s,N}(\zeta)| \leq C_2 \exp(s|\zeta|). \]

We put

\[ (3.6) \quad H_{<\nu>^s} = \{ f \in H^{(-\infty)}| \delta_{s,N}(D)f \in H_{<\nu>}, \quad 0 \leq \nu < N \} \]

and equip this space with the norm

\[ (3.7) \quad \|f\|_{<\nu>^s} = \|\delta_{s,N}(D)f\|_{<\nu>}, \quad 0 \leq \nu < N. \]

We can introduce in this space a Hilbert scalar product (to which the norm (3.7) corresponds):

\[ (3.8) \quad (f,g)_{<\nu>^s} = (\delta_{s,N}(D)f, \delta_{s,N}(D)g)_{<\nu>}, \quad 0 \leq \nu < N. \]

From (3.6) and Proposition 2.1 we can readily derive other equivalent definitions of the space \( H_{<\nu>^s} \). We have the following

**Proposition 3.1.** Let \( \nu > 0 \). Then the conditions below are equivalent for the Fourier hyperfunctions \( f \in H^{(-\infty)} \).

(i) \( \delta_{s,N}(D)f \in H_{<\nu>} \).

(ii') \( f \in \bigcap_{\nu'=1}^{n} H_{\nu'}^{(s)} \).

(iii') \( f \in \bigcap_{\Gamma \in \nu I} H_{\Gamma}^{(s)} \), and the norm

\[ (3.7') \quad \|f\|_{<\nu>^s} = \sup_{\Gamma \in \nu I} \|f\|_{\Gamma} \]
is finite.

(iii) The Fourier transform \( \hat{f} \in H_{<-\infty>} \) is continued holomorphically to the cube domain \( D_{\nu} \) and belongs to \( H_{<\nu>}^{(s)} \).
The condition (iii) implies that

\[ (3.9) \quad \mathcal{F} H^{(s)}_{(s)} = H^{(s)}_{(s)}. \]

Indeed, if \( f \in H^{(s)}_{(s)} \), then we have

\[
\| f \|^{(s)}_{(s)} = \sup_{\Gamma \in \nu I} \| \mathcal{F}^{(s)} f \| = \sup_{\Gamma \in \nu I} \| \exp(\langle x, \Gamma \rangle) \delta_{s,N}(D)f \| = \sup_{\Gamma \in \nu I} \| \delta_{s,N}(\xi + i\Gamma)\hat{f}(\xi + i\Gamma) \| = \| \hat{f} \|^{(s)}_{(s)}.
\]

**Proposition 3.2.** For \( s \geq s', \nu \geq \nu' \geq 0 \), we have

\[ (3.10) \quad H^{(s)}_{(s)} \subset H^{(s')}_{(s')} \]

i.e., \( \{ H^{(s)}_{(s)} \} \) is a scale of Banach (Hilbert) spaces.

**Proof.** If \( f \in H^{(s)}_{(s)} \), then it follows from Proposition 3.1 (i') that \( f = \delta_{-s,N}(D)g, \; g \in H_{(s')} \). Therefore we have

\[
\| f \|^{(s')}_{(s')} = \| \exp(\nu'|x|)\delta_{s',N}(D)f \| = \| \exp(\nu'|x|)\delta_{-(s-s'),N}(D)g \|
\leq \sum_{\kappa=1}^{2^n} \| \exp(\langle x, \nu' I(\kappa) \rangle)\delta_{-(s-s'),N}(D)g \|
= \sum_{\kappa=1}^{2^n} \| \delta_{-(s-s'),N}(\xi + i\nu' I(\kappa))\hat{g}(\xi + i\nu' I(\kappa)) \|
\leq C \sum_{\kappa=1}^{2^n} \| \hat{g}(\xi + i\nu' I(\kappa)) \| = C \sum_{\kappa=1}^{2^n} \| \exp(\langle x, \nu' I(\kappa) \rangle)g(x) \|
\leq 2^n C \| \hat{f} \|^{(s')}_{(s')}.
\]

\[ \square \]

We shall show that this scale is equivalent to Hölder’s scale.

**Proposition 3.3.** For every \( h, \nu > 0 \) we have the embeddings

\[ (3.11) \quad F_{(h,\nu+\epsilon)} \subset H^{(h/2)}_{(\nu)} \subset F_{(h/4,\nu)}, \; \epsilon > 0. \]
Proof. Let \( f \in F_{(h, \nu \epsilon)} \). Then we have

\[
\| f \|_{\nu, \epsilon}^{(h/2)} = \| \exp(\nu|x|) \delta_{h/2, N}(D)f \| \\
\leq \sum_{\kappa=1}^{2^n} \| \exp(<x, \nu I^{(\kappa)}>) \delta_{h/2, N}(D)f \| \\
= \sum_{\kappa=1}^{2^n} \| \delta_{h/2, N}(\xi + i
u I^{(\kappa)}) \hat{f}(\xi + i
u I^{(\kappa)}) \| \\
\leq C \sum_{\kappa=1}^{2^n} \| \exp(h/2|\xi + i
u I^{(\kappa)}|) \hat{f}(\xi + i
u I^{(\kappa)}) \| \\
\leq C \sum_{\kappa=1}^{2^n} \sum_{\alpha} \frac{(h/2)^{[\alpha]}}{\alpha!} \| (\xi + i
u I^{(\kappa)})^\alpha \hat{f}(\xi + i
u I^{(\kappa)}) \| \\
= C \sum_{\kappa=1}^{2^n} \sum_{\alpha} \frac{(h/2)^{[\alpha]}}{\alpha!} \| \exp(<x, \nu I^{(\kappa)}>) D^\alpha f \| \\
\leq C \sum_{\kappa=1}^{2^n} \sum_{\alpha} \frac{(h/2)^{[\alpha]}}{\alpha!} \| \exp(\nu|x|) D^\alpha f \| \\
\leq 2^n C \sum_{\alpha} \frac{(h/2)^{[\alpha]}}{\alpha!} \| f \|_{(h, \nu \epsilon)}^{(h/2)} \| \exp(-\epsilon|x|) \|. 
\]

This proves the left-hand embedding of (3.11).

To prove the right-hand embedding of (3.11), from Proposition 1.3 and 3.1 (ii) it suffices to show Sobolev’s embedding theorem written in the form

\[
(3.12) \quad H^{(h)}_{[\nu I^{(\kappa)}]} \subset F_{[h/2, \nu I^{(\kappa)}]}.
\]

Let \( f \in H^{(h)}_{[\nu I^{(\kappa)}]} \). Then we have

\[
\int |D^\gamma \hat{f}(\xi + i
u I^{(\kappa)})| d\xi \\
= \int |\delta_{h, N}(\xi + i
u I^{(\kappa)}) \hat{f}(\xi + i
u I^{(\kappa)})(\xi + i
u I^{(\kappa)})^\gamma \delta_{-h, N}(\xi + i
u I^{(\kappa)})| d\xi \\
\leq \| \delta_{h, N}(\xi + i
u I^{(\kappa)}) \hat{f}(\xi + i
u I^{(\kappa)}) \| \| (\xi + i
u I^{(\kappa)})^\gamma \delta_{-h, N}(\xi + i
u I^{(\kappa)}) \| 
\]
= \|\exp(<x,\nu I^{(s)}>)\delta_{h,N}(D)f\|\|((\xi + i\nu I^{(s)})^\gamma\delta_{-h,N}(\xi + i\nu I^{(s)}))\|
\leq C\gamma!(h/2)^{-|\gamma|}\|f\|_{[\nu I^{(s)}]}^{(h)}\|\exp(-\sqrt{2} - 1)h/2|\xi + i\nu I^{(s)}|)\|
\leq C'\gamma!(h/2)^{-|\gamma|}|f|_{[\nu I^{(s)}]}^{(h)}.

Since

\[ D^\gamma f(x) = (2\pi)^{-n/2}\int \exp(i < x, \xi + i\nu I^{(s)}) > \hat{D}^\gamma f(\xi + i\nu I^{(s)})d\xi, \]

we obtain

\[ |f|_{[h/2,\nu I^{(s)}]} \leq (2\pi)^{-n/2}C'|f|_{[\nu I^{(s)}]}^{(h)}. \]

From (3.11) it follows automatically that

\[ (3.13) \quad \mathcal{G} = \bigcap_{h,\nu} H^{(h)}_{<\nu>}. \]

§4. The space \( \mathcal{G}' \) as an inductive limit of Hilbert spaces

For \( \nu > 0 \) we define \( H^{(-s)}_{<\nu>} \), \( s \in \mathbb{R} \) as the Banach conjugate space of \( H^{(s)}_{<\nu>} \) and introduce in it the norm of a conjugate space:

\[ (4.1) \quad ||f||_{<-\nu>}^{(-s)} = \sup\{|(f, g)| \| g \|_{<-\nu>}^{(s)} \leq 1\}. \]

It follows from (2.2) that (4.1) is compatible with (2.1) for \( s = 0 \). Since \( \mathcal{G} \) is dense in \( H^{(s)}_{<\nu>} \), the functionals belonging to \( H^{(-s)}_{<\nu>} \) can be regarded as elements of \( \mathcal{G}' \).

Note that \( H^{(h/2)}_{<-\nu>} \subset H_{<-\nu>} \) and \( F_{(h, -\nu + \epsilon)} \subset H_{<-\nu>} \). Therefore each function \( f \in F_{(h, -\nu + \epsilon)} \), \( \epsilon > 0 \), belongs to \( H^{(-h/2)}_{<-\nu>} \).

Proposition 4.1. For \( h, \nu > 0 \) we have

\[ (4.2) \quad F_{(h, -\nu + \epsilon)} \subset H^{(h/2)}_{<-\nu>}. \]
Proof. Let \( f \in F_{(h,-\nu+\epsilon)} \) and \( g \in H_{<\nu>^{(-h/2)}} \). Then it follows from Proposition 3.1 (i') that \( g = \delta_{h/2,N}(D)g_0, \ g_0 \in H_{<\nu>} \). We construct a system of functions \( \chi^{(\kappa)}, \ \kappa = 1, \ldots, 2^n \), \( \chi^{(\kappa)} \geq 0 \), possessing the following properties:

(a) \( \sum_{\kappa=1}^{2^n} \chi^{(\kappa)}(x) = 1 \)

(b) If \( f \in F_{(h,-\nu+\epsilon)} \), then \( \chi^{(\kappa)}f \in F_{[h,(-\nu+\epsilon)} \), and

\[
|\chi^{(\kappa)}f|_{[h,(-\nu+\epsilon)} \leq \text{const}|f|_{(h,-\nu+\epsilon)}.
\]

If \( x_i, \ldots, x_k \geq 0 \) and \( x_{k+1}, \ldots, x_n < 0 \), let \((\epsilon_1, \ldots, \epsilon_n)\) be the coordinates of a vertex \( I^{(\kappa)} \), where \( \epsilon_i = \cdots = \epsilon_k = 1 \) and \( \epsilon_{k+1} = \cdots = \epsilon_n = -1 \). Then we put

\[
\chi^{(\kappa)}(x) = \prod_{i=1}^{k} \exp(-x_i^2) \prod_{i=k+1}^{n} (1 - \exp(-x_i^2)).
\]

It is obvious that (a) is fulfilled.

Since \( <I^{(\kappa)}, x> = \sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} x_i = |x| \), it follows from the proof of Theorem 1.1 that (b) holds. Then we have

\[
\begin{align*}
|\langle f, g \rangle| & \leq \sum_{\kappa=1}^{2^n} |\langle \chi^{(\kappa)}f, g \rangle| \\
& = \sum_{\kappa=1}^{2^n} |\langle \exp(-\nu|x|)\delta_{h/2,N}(D)\chi^{(\kappa)}f, \exp(\nu|x|)g_0 \rangle| \\
& \leq \sum_{\kappa=1}^{2^n} \| \exp(-\nu I^{(\kappa)}x) \delta_{h/2,N}(D) \chi^{(\kappa)}f \| \| \exp(\nu|x|)g_0 \| \\
& \leq \sum_{\kappa=1}^{2^n} \| \delta_{h/2,N}(\xi - i\nu I^{(\kappa)})\chi^{(\kappa)}f(\xi - i\nu I^{(\kappa)}) \| \| g \|_{<\nu>^{(-h/2)}} \\
& \leq C \sum_{\kappa=1}^{2^n} \| \exp(h/2|\xi - i\nu I^{(\kappa)}|)\chi^{(\kappa)}f(\xi - i\nu I^{(\kappa)}) \| \| g \|_{<\nu>^{(-h/2)}} \\
& \leq C \sum_{\kappa=1}^{2^n} \sum_{\alpha} \frac{(h/2)|\alpha|}{\alpha!} \| (\xi - i\nu I^{(\kappa)})^\alpha \chi^{(\kappa)}f(\xi - i\nu I^{(\kappa)}) \| \| g \|_{<\nu>^{(-h/2)}} \\
& \leq C \sum_{\kappa=1}^{2^n} \sum_{\alpha} \frac{(h/2)|\alpha|}{\alpha!} \| \exp(-\nu I^{(\kappa)}x)D^\alpha(\chi^{(\kappa)}f)(x) \| \| g \|_{<\nu>^{(-h/2)}}
\end{align*}
\]
\[ \leq C \sum_{\kappa=1}^{2^n} \sum_{\alpha} \frac{(h/2)^{|\alpha|}}{\alpha!} |\chi^{(\kappa)} f|_{[h, (-\nu+\epsilon), I^{(\kappa)}]} h^{-|\alpha|} \alpha! \| \exp(-\epsilon |x|)\| g \|^{(-h/2)} \]

\[ \leq C |f|_{[h, (-\nu+\epsilon)]} \| g \|^{(-h/2)} , \]

whence it follows. \hfill \square

Thus, we have
\[ (4.3) \quad G \subset F_{(h, -\nu+\epsilon)} \subset H_{\leq h}^{(-h/2)} \subset G' \forall h, \nu > 0, \]

where all the embeddings are dense.

**Proposition 4.2.** For \( h, \nu > 0 \), we have
\[ (4.4) \quad H_{\leq h}^{(h/2)} \subset F_{(h/4, -\nu)}. \]

**Proof.** Let \( f \in H_{\leq h}^{(h/2)} \). Since
\[ D^\alpha f(x) = (2\pi)^{-n/2} \int_{Im\zeta = \omega_j} \exp(i < x, \zeta >) \zeta^\alpha \hat{f}(\zeta) d\zeta, \quad \omega_j = -\nu \frac{|x_j|}{x_j} \]
\[ = (2\pi)^{-n/2} \int_{Im\zeta = \omega_j} \exp(i < x, \zeta >) \delta_{-h/2, N}(\zeta) \zeta^\alpha \delta_{h/2, N}(\zeta) \hat{f}(\zeta) d\zeta, \]

we have
\[ |D^\alpha f(x)| \leq (2\pi)^{-n/2} \exp(\nu |x|) \| \delta_{-h/2, N}(\zeta) \zeta^\alpha \| \| \delta_{h/2, N}(\zeta) \hat{f}(\zeta) \| \]
\[ \leq (2\pi)^{-n/2} \exp(\nu |x|) \| \exp(-\frac{h}{2\sqrt{2}} |\zeta|) \zeta^\alpha \| \]
\[ \times \| \exp(< x, Im\zeta >) \delta_{h/2, N}(D) f(x) \| \]
\[ \leq (2\pi)^{-n/2} \alpha! \left( \frac{h}{2\sqrt{3}} \right)^{-|\alpha|} \exp(\nu |x|) \| \exp(-\frac{h}{2\sqrt{2}} - \frac{1}{\sqrt{3}} |\zeta|) \| \| f \|^{(h/2)}_{\leq h} \]
\[ \leq (2\pi)^{-n/2} \alpha! \left( \frac{h}{4} \right)^{-|\alpha|} \exp(\nu |x|) \| \exp(-\frac{h}{2\sqrt{2}} - \frac{1}{\sqrt{3}} |\zeta|) \| \| f \|^{(h/2)}_{\leq h} , \]

whence it follows. \hfill \square
Since the spaces $H^{(s)}_{<\nu>}$, $\nu \geq 0$, form a scale, the spaces $H^{(-s)}_{<\nu>}$ form the dual scale, and we can consider the inductive limits $H^{(-\infty)}_{<\nu>}$, $H^{(-\infty)}_{<\nu>}$, and endow them with the natural topology. By virtue of the reflexivity of $H^{(s)}_{<\nu>}$, these limits are regular, and, according to the general properties of regular inductive limits, we have the topological isomorphisms

\[(4.5) \quad (H^{(\infty)}_{<\nu>})' = H^{(-\infty)}_{<-\nu>} , \]

\[(4.6) \quad \mathcal{G}' = \bigcup_{s, \nu} H^{(s)}_{<\nu>} , \]

where the left-hand and right-hand spaces are equipped with topologies of strong conjugate space and inductive limits, respectively.

In view of the duality (4.5), we can define PDO on $H^{(-\infty)}_{<-\nu>}$. Let $a(\zeta) \in F^{(N, -\infty)}$, and for $N > \nu$ we put

\[(4.7) \quad (a(D)f, g) = (f, a(-D)g), \quad \forall g \in H^{(\infty)}_{<-\nu>} . \]

On the dense subset $\mathcal{G} \subset H^{(-s)}_{<-\nu>}$ this definition is compatible with the one stated in (3.4). Proposition 3.1 (i') and (2.2) imply the isomorphism

\[(4.8) \quad \delta_{-s,N}(D)H^{(-s)}_{<-\nu>} = H_{<-\nu>} \]

with the equality of norms:

\[(4.1') \quad \|f\|^{(-s)}_{<-\nu>} = \|\delta_{-s,N}(D)f\|_{<-\nu>} , \]

Proposition 2.1 (ii) and (4.8) imply the following

**PROPOSITION 4.3.** For $\nu > 0$ and any $s \in \mathbb{R}$ we have

\[(4.9) \quad H^{(-s)}_{<-\nu>} = \sum_{\kappa=1}^{2^n} H^{(-s)}_{[-\nu I(\kappa)]} , \]

and (4.1) is equivalent to the norm

\[(4.1'') \quad \inf_{f=f_1+\cdots+f_n} \sum_{\kappa=1}^{2^n} \|f_\kappa\|^{(-s)}_{[-\nu I(\kappa)]} . \]
of the right-hand space of (4.9).

(3.11), (4.3) and Proposition 4.2 implies

\[ \mathcal{O} = \bigcup_{\nu} H_{<\nu>}^{(\infty)}, \quad \mathcal{O}' = \bigcap_{\nu} H_{<\nu>}^{(-\infty)}, \]

(4.10)
\[ \mathcal{M} = \bigcap_{s} H_{<\nu>}^{(s)} \]

Denote by \( F_{<\nu>}^{(s)} \) the space of holomorphic functions in the tube domain \( D_\nu \) and having a finite norm

\[ |\psi|_{(s)}^{<\nu>} = \sup_{\zeta \in D_\nu} |\delta_{s,N}(\zeta)||f(\zeta)|. \]

It follows from Proposition 1.2 and (3.5) that if \( s \geq 0 \), then

\[ F_{(s)}^{(\nu,s)} \simeq F_{(s)}^{<\nu>} \subset F_{(s)}^{(\nu,s/\sqrt{2})} \simeq F_{(s)}^{(\nu,s/\sqrt{2})}, \]

(4.12)

and hence (1.2) implies that

(1.2') \[ \mathcal{M} = \bigcap_{\nu} F_{(-\infty)}^{<\nu>}. \]

**Proposition 4.4.** The following isomorphisms of vector spaces hold:

\[ \mathcal{F} \mathcal{O}' = \mathcal{M}, \]

(4.13)

where \( \mathcal{F} \) is a Fourier-Laplace operator.

**Proof.** By (3.9) and (4.10) we obtain

\[ \mathcal{F} \mathcal{O}' = \bigcup_{\nu} \bigcup_{s} \mathcal{F} H_{<\nu>}^{(s)} = \bigcup_{\nu} \bigcup_{s} H_{<\nu>}^{(s)} = \bigcap_{\nu} H_{<\nu>}^{(-\infty)}. \]

Hence, the proof of (4.13) reduces to the proof of the equivalence of the scales \( \{ F_{(s)}^{<\nu>} \} \) and \( \{ H_{(s)}^{<\nu>} \} \) where \( \nu > 0 \). We shall prove the following. □

**Proposition 4.5.** For every \( \nu > 0 \), the embeddings

\[ F_{(s+\epsilon)}^{<\nu>} \subset H_{(s)}^{<\nu>} \subset F_{(s-\epsilon')}^{<\nu>}, \quad \forall \epsilon, \epsilon' > 0, \]

(4.14)

take place.
**Proof.** If \( N > \nu \), then the operator of multiplication by the function \( \delta_{r,N}(\zeta) \) generates the isomorphisms

\[
F^<\nu>_{(s)} \to F^<\nu>_{(s-r)}, \quad H^<\nu>_{(s)} \to H^<\nu>_{(s-r)} \quad (\psi \to \delta_{r,N}(\zeta)\psi).
\]

Hence, when proving (4.14) we can confine ourselves to the case of \( s > \epsilon' \).

The left-side inclusion in (4.14) is obvious. According to (3.9), (1.5), (4.12) and Proposition 3.3,

\[
H^<\nu>_{(s)} = \mathcal{F}H^<\nu>_{(s/2,\nu)} \subset F^{(\nu,s/2)}_{(s/2)} \subset F^<\nu>_{(s/2)}.
\]

This proves the proposition. \( \square \)

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