1. Introduction

Delta sequences play an important role in convergence and approximation theory[10]. These sequences arise in the kernel approximation method, in orthogonal series, approximation in integral transform and in polynomial approximations[3,6,7].

Most of the results related to Gibbs’ phenomenon are one dimensional. Yet the phenomenon is most troublesome in higher dimensional applications, in particular image analysis, where it shows up or rings around discontinuities. This phenomenon has been noticed by [2,5] for at least a century, but is usually (inaccurately) attributed to Gibbs[1]. It appears in most approximations involving partial sums of orthogonal series[4,8,9,10]. It refers to the overshoot of the partial sum approximation near jump discontinuity which does not disappear with higher order partial sums.

2. Convolution type delta sequence
Definition 1. A delta sequence, $\delta_m(x, y)$, $x, y \in \Omega \subset \mathbb{R}^d$, is a sequence or functions in $L^\infty(\Omega \times \Omega)$ which converge to the delta function i.e.,

$$\int_\Omega \delta_m(x, y)\phi(y)dy \longrightarrow \phi(x) \quad \text{as} \quad m \to \infty$$

for all $\phi \in C^\infty_c(\Omega)$, $y \in \Omega$.

We shall generally assume that one of the variables is a parameter so that $\delta_m(x, y)$ is considered as a function in $L^\infty(\mathbb{R}^d)$ for each fixed $x \in \mathbb{R}^d$.

Definition 2. A delta sequence is of convolution type if

$$\delta_m(x, y) = m^dK(m(x - y)),$$

where $K \in L^1(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} K(x)dx = 1$.

Hereafter we mean $\delta_m * f(x)$ the approximation given by

$$\int_{\mathbb{R}^d} m^dK(m(x - y))f(y)dy = \int_{\mathbb{R}^d} m^dK(m(y))f(x - y)dy.$$

We have several convergence theorems related with the approximation by a convolution type delta sequence.

Theorem 1. Suppose $f \in L^\infty(\mathbb{R}^d)$ satisfies a uniform Lipschitz condition. Then we have $\delta_m * f(x) \to f(x)$ uniformly on $\mathbb{R}^d$ as $m \to \infty$.

Proof. By definition, we have

$$\delta_m * f(x) - f(x) = \int_{\mathbb{R}^d} \delta_m(y) [f(x - y) - f(x)] dy$$

$$= \int_{\mathbb{R}^d} m^dK(my) [f(x - y) - f(x)] dy$$

$$= \int_{\mathbb{R}^d} K(mt) \left[ f(x - \frac{t}{m}) - f(x) \right] dt.$$

By a uniform Lipschitz condition, there exists a constant $C > 0$ such that

$$\left| f \left( x - \frac{t}{m} \right) - f(x) \right| \leq \frac{C}{m} |t|,$$

which converges 0 as $m$ approaches infinity.
Therefore, by the Lebesgue’s dominant convergence theorem, we have
\[ |\delta_m * f(x) - f(x)| \leq \int_{\mathbb{R}^d} |K(t)| \left| f \left( x - \frac{t}{m} \right) - f(x) \right| dt \to 0 \quad \text{as} \quad m \to \infty, \]
which means that for given \( \epsilon > 0 \) there exists \( m_0 \in \mathbb{N} \) such
\[ |\delta_m * f(x) - f(x)| < \epsilon \quad \text{for all} \quad m \geq m_0. \]
\[ \square \]
Under weaker condition, we can have convergence theorem whose proof is a little different from the stronger condition.

**Theorem 2.** Suppose \( f \in L^\infty(\mathbb{R}^d) \) is continuous on \( \mathbb{R}^d \). Then we have
\[ \delta_m * f(x) \to f(x) \quad \text{uniformly on} \quad \mathbb{R}^d \quad \text{as} \quad m \to \infty. \]

**Proof.** Let \( x \in \mathbb{R}^d \) and \( \epsilon > 0 \). Then there exists \( \eta > 0 \) such that
\[ |f(x) - f(y)| < \frac{\epsilon}{2||K||_1} \quad \text{for all} \quad y, \ |x - y| < \eta. \]
We consider
\[ \delta_m * f(x) - f(x) = \int_{\mathbb{R}^d} m^d K(my) \{ f(x - y) - f(x) \} dy \]
\[ = \left( \int_{|y| < \eta} + \int_{|y| \geq \eta} \right) (m^d K(my) \{ f(x - y) - f(x) \}) dy \]
\[ = I_1 + I_2. \]
We estimate \( I_1 \) and \( I_2 \) as follows;
\[ |I_1| \leq \int_{|y| < \eta} m^d K(my) \{ f(x - y) - f(x) \} dy \]
\[ < \frac{\epsilon}{2||K||_1} \int_{|y| < \eta} m^d K(my) \]
\[ = \frac{\epsilon}{2||K||_1} \int_{|t| < m\eta} K(t)dt \]
\[ \leq \frac{\epsilon}{2||K||_1} \int_{\mathbb{R}^d} |K(t)|dt = \frac{\epsilon}{2}. \]
For $I_2$, we have
\[
|I_2| \leq 2||f||_{\infty} \int_{|y| \geq \eta} m^d |K(my)| dy \\
= 2||f||_{\infty} \int_{|t| \geq m\eta} |K(t)| dt.
\]
Since $\int_{|t| \geq m\eta} |K(t)| dt \to 0$ as $m \to \infty$, there exists $m_0 \in \mathbb{N}$ such that
\[
\int_{|t| \geq m\eta} |K(t)| dt < \frac{\epsilon}{4||f||_{\infty}} \quad \text{for all} \quad m \geq m_0.
\]
Combining $I_1$ and $I_2$ for $m \geq m_0$, we have
\[
|\delta_m * f(x) - f(x)| < \epsilon \quad \text{for all} \quad x \in \mathbb{R}^d.
\]
\[\square\]

As a result of the theorem 2, the following property holds. But we give a full proof in the interest of completeness.

**Corollary 3.** Suppose $f \in L^{\infty}(\mathbb{R}^d)$ is continuous at $0$ and $x_m \to 0$ as $m \to \infty$. Then we have
\[
\delta_m * f(x_m) \to f(0) \quad \text{as} \quad m \to \infty.
\]

**Proof.** Given $\epsilon > 0$, there exists $\eta > 0$ such that
\[
|x| < \eta \quad \Rightarrow \quad |f(x) - f(0)| < \frac{\epsilon}{2||f||_{1}}. \quad (*)
\]
Now we consider the following equalities:
\[
\delta_m * f(x_m) - f(0) = \int_{\mathbb{R}^d} m^d K(my) \{f(x_m - y) - f(0)\} dy \\
= \left( \int_{|y| < \frac{\eta}{2}} + \int_{|y| \geq \frac{\eta}{2}} \right) m^d K(my) \{f(x - y) - f(x)\} dy \\
= I_3 + I_4.
\]
For $I_3$, we have
\[ |I_3| \leq \int_{|y| < \frac{\eta}{2}} \frac{m^d |K(my)||f(x_m - y) - f(0)|}{m^d |K(My)|} dy. \]

Since \( x_m \to 0 \) as \( m \to \infty \), there exists \( m_0 \in \mathbb{N} \) such that
\[
m > m_0 \Rightarrow |x_m| < \frac{\eta}{2} \quad \text{and so that} \quad |x_m - y| < \eta.
\]

By (*), we obtain
\[
|I_3| < \frac{\epsilon}{2||f||_1} \int_{|y| < \frac{\eta}{2}} m^d |K(my)| dy
\]
\[
= \frac{\epsilon}{2||f||_1} \int_{|t| < \frac{\eta}{2m}} |K(t)| dt
\]
\[
< \frac{\epsilon}{2||f||_1} ||K||_1 = \frac{\epsilon}{2}.
\]

For \( I_4 \), we have
\[
|I_4| \leq 2||f||_{\infty} \int_{|y| \geq \frac{\eta}{2}} m^d |K(my)| dy
\]
\[
= 2||f||_{\infty} \int_{|t| \geq \frac{\eta}{2m}} |K(t)| dt.
\]

Noticing \( \int_{|t| \geq \frac{\eta}{2m}} |K(t)| dt \to 0 \) as \( m \to \infty \), there exists \( m_1 \in \mathbb{N} \) such that
\[
\int_{|t| \geq \frac{\eta}{2m}} |K(t)| dt < \frac{\epsilon}{4||f||_{\infty}} \quad \text{for} \quad m \geq m_1,
\]
which means \( |I_4| < \frac{\epsilon}{2} \).

Finally taking \( m^* = \max\{m_0, m_1\} \), we obtain
\[
|\delta_m * f(x_m) - f(0)| < \epsilon \quad \text{for all} \quad m \geq m^*.
\]

\[ \square \]

3. Gibbs’ phenomenon in higher dimension

To study Gibbs phenomenon (Gibbs an abbreviation) in higher dimension, we need to define the concept of jump discontinuity and redefine the definition of Gibbs. We shall assume that the functions are piecewise continuous, that is, continuous on \( \mathbb{R}^d \) except for a \( d - 1 \) dimensional hypersurface. In order to avoid pathology, we assume the hypersurface to be smooth and to be given by a function \( g(x) = c \), where \( g \in C^1(\mathbb{R}^d) \).
Definition 3. Suppose \( g(x_0) = c \) and let \( \gamma \) be given by the normalized gradient
\[
\gamma = \frac{\nabla g(x_0)}{|\nabla g(x_0)|}
\]

Then we say function \( f \in \mathbb{R}^d \) has jump discontinuity at \( x_0 \) in the direction \( \gamma \) if the following condition holds;
\[
\lim_{0 < t \to 0} f(x_0 + t\gamma) = f(x_0; \gamma^+) > f(x_0; \gamma^-) = \lim_{0 < t \to 0} f(x_0 - t\gamma).
\]
Otherwise we take its negative.

Examples. (1) Let \( f(x, y) = (x + 1)H(2x - x^2 - y^2) \). This function has a jump discontinuity along the circle \( (x - 1)^2 + y^2 = 1 \). The gradient (normalized) of \( g(x, y) = (x - 1)^2 + y^2 = 1 \) is given by
\[
\frac{\nabla g(x_0)}{|\nabla g(x_0)|} = \frac{(2x - 2, 2y)}{2\sqrt{(x - 1)^2 + y^2}} = (x - 1, y).
\]
If we are interested in the point \( x_0 = (0, 0) \), then \( \gamma = (-1, 0) \) and
\[
f(0 + t\gamma) = f(-t, 0) = (-t + 1)h(-2t - t^2) = \begin{cases} 
0, & -2 \leq t \leq 0 \\
-t + 1, & \text{otherwise}
\end{cases}
\]
We \( f(0; \gamma^+) = 0, f(0; \gamma^-) = 1 \) and \( f(\gamma^+) < f(\gamma^-) \). So we change the direction of \( \gamma \) and replace it as \( -\gamma \).

(2) Let \( f(x) = H(n \cdot x) \), where \( n \) is a unit vector in \( \mathbb{R}^d \), \( n \cdot x \) is inner product and \( H(t) \) is the Heaviside function. Then \( f \) take the value 1 on a half space and 0 on its complement. It has jump discontinuity in the direction \( n \).

Definition 4. Let \( f \in L^2(\mathbb{R}^d) \) be piecewise continuous with jump discontinuity at \( x_0 \) in the direction \( \gamma \). Then the approximation \( f_m(x) = \delta_m * f(x) \) is said to exhibit Gibbs phenomenon at \( x_0 \) in the direction \( \gamma \) if there exist \( \mu \) with \( \gamma \cdot \mu > 0 \) and a sequence \( 0 < a_m \to 0 \) such that \( f_m(a_m\mu) \) converges to a number \( \lambda(\gamma) > f(x_0; \gamma^+) \) as \( m \to \infty \).

Lemma 4. Let \( f(x) = H(n \cdot x) \). The approximation \( f_m(x) = \delta_m * f(x) \) exhibits Gibbs at \( x = 0 \) in the direction \( n \) if and only if there exist \( a > 0 \) such that
\[
\int_{n \cdot x > a} K(x)dx < 0.
\]
Proof. We suppose $f_m(x)$ exhibits Gibbs at $0$ in the direction $n$. Then there exist $\mu$ with $n \cdot \mu > 0$ and a sequence $0 < a_m \to 0$ such that

$$\lim_{m \to \infty} f_m(a_m \mu) > f(0; n^+) = 1.$$ 

Since $f_m(a_m \mu)$ is given by

$$f_m(a_m \mu) = \int_{\mathbb{R}^d} m^d K(my) f(a_m \mu - y) dy$$

$$= \int_{a_m \mu \cdot n \geq y \cdot n} m^d K(my) dy$$

$$= \int_{a_m \mu \cdot n \geq y \cdot n} K(t) dt.$$ 

From $\lim_{m \to \infty} f_m(a_m \mu) > 1$, there is $m_0 \in \mathbb{N}$ such that $f_{m_0}(a_{m_0} \mu) > 1$. So by letting $m_0 a_{m_0} \mu \cdot n = a$, we have $a > 0$ and

$$\int_{t \leq a} K(t) dt = 1 - \int_{t > a} K(t) dt > 1.$$ 

Therefore we have

$$\int_{n \cdot x > a} K(x) dx < 0.$$ 

To show the other direction, we suppose the following holds

$$\int_{n \cdot x > a} K(x) dx < 0 \quad \text{for} \quad a > 0.$$ 

From the approximation $f_m(x)$ given by

$$f_m(x) = \int_{\mathbb{R}^d} m^d K(my) f(x - y) dy$$

$$= \int_{y \cdot n \leq y \cdot n} m^d K(my) dy$$

$$= \int_{\frac{a}{m} \cdot n \leq x \cdot n} K(t) dt,$$

we take a sequence $x_m = a_m \gamma = a_m \gamma$ with $\gamma \cdot n > 0$. Then $0 < a_m \to 0$ and

$$f_m(x_m) = f_m(a_m \gamma) = \int_{t \leq a} K(t) dt.$$ 

By letting $\gamma = n$, we have

$$f_m(a_m n) = \int_{t \leq a} K(t) dt = 1 - \int_{t \leq a} K(t) dt.$$
Since \( \int_{t}^{n} K(t) \, dt < 0 \), we have \( f_{m}(a_{m}n) > 1 \) and \( \lim_{m \to \infty} f_{m}(a_{m}n) > 1 \). Therefore \( f_{m}(x) \) exhibits Gibbs at \( 0 \) in the direction \( n \). \( \square \)

We try to show Gibbs in general functions.

**Theorem 5.** Suppose \( g \) be a piecewise continuous with jump at \( 0 \) in the direction \( \gamma \). Then the approximation \( g_{m}(x) \) exhibits Gibbs at \( 0 \) in the direction \( \gamma \).

**Proof.** We suppose the jump in the direction \( \gamma \) is \( \alpha > 0 \), i.e.,

\[
g(0; \gamma^{+}) - g(0; \gamma^{-}) .
\]

We define \( q \) by

\[
q(x) = \begin{cases} 
  g(x) - \alpha H(\gamma \cdot x), & x \neq 0 \\
  g(0; \gamma^{+}), & x = 0.
\end{cases}
\]

Then \( q(x) \) is continuous at \( 0 \) in the direction \( \gamma \) since

\[
\lim_{0 < t \to 0} q(t\gamma) = g(0; \gamma^{+}) - \alpha = f(0; \gamma^{-}) \quad \text{and} \quad \lim_{0 > t \to 0} q(t\gamma) = g(0; \gamma^{-}).
\]

Since \( H(\gamma \cdot x) \) exhibits Gibbs at \( 0 \) in the direction \( \gamma \), there exist \( \mu \) with \( \mu \cdot \gamma > 0 \) and a sequence \( 0 < a_{m} \to 0 \) such that \( \lim_{m \to \infty} H_{m}(a_{m}\mu \cdot \gamma) > 1 \). Then we have

\[
g_{m}(a_{m}\mu) = g_{m}(a_{m}\mu) - \alpha H_{m}(a_{m}\mu \cdot \gamma).
\]

By theorem 2 and theorem 3, we have

\[
\lim_{m \to \infty} g_{m}(a_{m}\mu) = \lim_{m \to \infty} g_{m}(a_{m}\mu) - \lim_{m \to \infty} \alpha H_{m}(a_{m}\mu \cdot \gamma) = g(0; \gamma^{-}),
\]

and so

\[
\lim_{m \to \infty} g_{m}(a_{m}\mu) = \alpha \lambda(\gamma) + g(0; \gamma^{+}),
\]

where \( \lambda(\gamma) > 1 \). Considering the following difference

\[
\{ \alpha \lambda(\gamma) + g(0; \gamma^{-}) \} - g(0; \gamma^{+}) = (\lambda - 1)\alpha > 0,
\]

we conclude that \( g_{m}(a_{m}\mu) > g(0; \gamma^{+}) \), which means \( g_{m}(x) \) exhibits Gibbs at \( 0 \) in the direction \( \gamma \). \( \square \)
Remark. Once we show the existence of Gibbs at 0, we can show the existence at $x_0$ by translation. So it’s enough to investigate the behavior at 0 to study Gibbs.

Acknowledgement

The author would like to thank Professor Gilbert G. Walter for several discussions that motivate this work.

References

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