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RECENT PROGRESS IN ASYMPTOTIC EXPANSION METHODS TO IDENTIFY SMALL INCLUSIONS

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ABSTRACT. We briefly explain recent methods in inverse problems to detect small electric and elastic inclusions. The methods rely on asymptotic expansion of the perturbation due to presence of inclusions. The asymptotic expansion is expressed by polarization tensors.

Because of its practical importance, inverse problems to identify unknown inclusions in a conductor or an elastic body by boundary measurements attract much attention lately. Among several methods to identify inclusions, the method of asymptotic expansions turns out to be very effective to identify the location and polarization tensors [1, 2, 4, 5, 6, 7, 9, 10, 12, 11, 13, 15, 16, 17, 20]. In this paper, we briefly summarize recent progress on the asymptotic methods.

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with a connected Lipschitz boundary $\partial\Omega$. Let ν denote the unit outward normal to $\partial\Omega$. Suppose that Ω contains a small inhomogeneity D of the form $D = z + \epsilon B$, where B is a bounded Lipschitz domain in $\mathbb{R}^d$ containing the origin. We assume that the domains D is well separated apart from each other and apart from the boundary. More precisely, we assume that there exists a constant $c_0 > 0$ such that

$$\operatorname{dist}(z, \partial\Omega) \geq c_0 > 0, \tag{1}$$

and ϵ, the common order of magnitude of the diameters of the inhomogeneities, is sufficiently small. We also assume that the "background" is homogeneous with conductivity 1 and the inhomogeneities D have conductivities k.

Let u_ϵ denote the steady-state voltage potential in the presence of the conductivity inhomogeneity, *i.e.*, the solution to

$$\begin{cases} \nabla \cdot \Big(\chi(\Omega \setminus \overline{D}) + k\chi(D) \Big) \nabla u_\epsilon = 0 \quad \text{in } \Omega, \\ \dfrac{\partial u_\epsilon}{\partial \nu}\Big|_{\partial\Omega} = g. \end{cases} \tag{2}$$

Let U denote the "background" potential, that is, the solution to

$$\Delta U = 0 \quad \text{in } \Omega, \quad \frac{\partial U}{\partial \nu}\Big|_{\partial\Omega} = g. \tag{3}$$

The function g represents the applied boundary current; it belongs to $L^2_0(\partial\Omega) = \{g \in L^2(\partial\Omega), \int_{\partial\Omega} g = 0\}$. The potentials, u_ϵ and U, are normalized by $\int_{\partial\Omega} u_\epsilon = \int_{\partial\Omega} U = 0$. The following expansion of u_ϵ as $\epsilon \to 0$ was derived in [2].

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Theorem 1. *The following pointwise asymptotic expansion on $\partial\Omega$ holds for $d = 2, 3$:*

$$u_\epsilon(x) = U(x) - \epsilon^d \sum_{|\alpha|=1}^{d} \sum_{|\beta|=1}^{d-|\alpha|+1} \frac{\epsilon^{|\alpha|+|\beta|-2}}{\alpha!\beta!} \partial^\alpha U(z) m_{\alpha\beta} \partial_z^\beta N(x,z) + O(\epsilon^{2d}), \tag{4}$$

where the remainder $O(\epsilon^{d+n})$ is dominated by $C\epsilon^{d+n}\|g\|_{L^2(\partial\Omega)}$ for some C independent of $x \in \partial\Omega$. Here $N(x,z)$ is the Neumann function for the Laplacian, and $M_{\alpha\beta}$, $\alpha, \beta \in \mathbb{N}^d$, are the generalized polarization tensors defined later.

In fact, a more complete asymptotic expansion than (4) was derived in [2]. The expansion formula describes in a precise manner how the presence of inclusions perturb the solution, and will play a crucial role in detecting inclusions. The leading order term in (4) was derived in [12]; see also [13] for the case of perfectly conducting or insulating inhomogeneities. If there are finitely many well separated inclusions D_l, $l = 1, ..., m$, then by taking summation over l, we can derive an asymptotic formula up to the error $O(\epsilon^{2d})$.

The generalized polarization tensors (GPT) $m_{\alpha,\beta}$, $\alpha = (\alpha_1, \ldots, \alpha_d), \beta = (\beta_1, \cdots, \beta_d) \in \mathbb{N}^d$, are defined to be

$$m_{\alpha\beta} := \int_{\partial B} y^\beta (\lambda I - \mathcal{K}_B^*)^{-1} (\nu_y \cdot \nabla y^\alpha)(y) d\sigma(y), \quad \lambda = \frac{k+1}{2(k-1)}.$$

See [2]. This concept of GPT generalizes those by Pólya, Szegö, and Schiffer [19, 18]. Some important properties of GPT such as symmetry and positivity were proved in [3]. The same properties for the first order tensor were obtained in [12]. It is also proved in [3] that the full knowledge of GPT on B determines the Dirichlet-to-Neumann map, and hence k and B completely by a result in [14].

If $|\alpha| = |\beta| = 1$ and $\sum_{\alpha \in I} a_\alpha^2 = 1$, then positivity estimates of GPT takes the following form:

$$\frac{|k-1|}{k+1}|B| \leq \Big|\sum_{\alpha,\beta \in I} a_\alpha a_\beta m_{\alpha\beta}\Big| \leq C|B|. \tag{5}$$

We note that the estimates (5) are not optimal. If B is a three dimensional ball, then

$$\Big|\sum_{\alpha,\beta \in I} a_\alpha a_\beta m_{\alpha\beta}\Big| = \frac{3|k-1|}{k+2}|B|.$$

It is conjectured by Pólya-Szegö that for any domain B in $\mathbb{R}^3$

$$\Big|\sum_{\alpha,\beta \in I} a_\alpha a_\beta m_{\alpha\beta}\Big| \geq \frac{3|k-1|}{k+2}|B|, \tag{6}$$

which implies that among all domains of the same volume the polarization tensor of the ball has the minimal trace [18]. This conjecture remains unproven.

Let us now explain how the asymptotic formula (4) can be used to identify the unknown inclusion D. Here we assume that the conductivity k is known. The inverse problem we consider here is to identity the location and size of the unknown inclusion D by means of the applied current and voltage measured on $\partial\Omega$. To be more precise, for a given current g let $f := u|_{\partial\Omega}$ where u is the solution of (2). The problem is to identify D using finitely many pairs of (f, g).

For a given voltage-current pair (f, g), which is measured on $\partial\Omega$, define a function $H[g]$ by

$$H[g](x) = -\mathcal{S}_\Omega(g)(x) + \mathcal{D}_\Omega(u_\epsilon|_{\partial\Omega})(x), \quad x \in \mathbb{R}^d \setminus \overline{\Omega}, \tag{7}$$

where $\mathcal{S}_\Omega$ and $\mathcal{D}_\Omega$ denote the single and double layer potentials on Ω: Γ is the fundamental solution for the Laplacian and

$$\mathcal{S}_\Omega\phi(x) := \int_{\partial\Omega} \Gamma(x-y)\phi(y)d\sigma(y), \quad x \in \mathbb{R}^d,$$

$$\mathcal{D}_\Omega\phi(x) := \int_{\partial\Omega} \frac{\partial}{\partial\nu_y}\Gamma(x-y)\phi(y)d\sigma(y), \quad x \in \mathbb{R}^d \setminus \partial\Omega.$$

Substituting (4) into (7), one can derive the following asymptotic expansion of $H[g]$ outside Ω: for $x \in \mathbb{R}^d \setminus \overline{\Omega}$,

$$H[g](x) = -\epsilon^d \sum_{|\alpha|=1}^{d} \sum_{|\beta|=1}^{d-|\alpha|+1} \frac{\epsilon^{|\alpha|+|\beta|-2}}{\alpha!\beta!}\partial^\alpha U(z) m_{\alpha\beta} \partial_z^\beta \Gamma(x,z) + O(\frac{\epsilon^{2d}}{|x|^{d-1}}). \tag{8}$$

We note that the expansion (8) was derived when U is linear in [8].

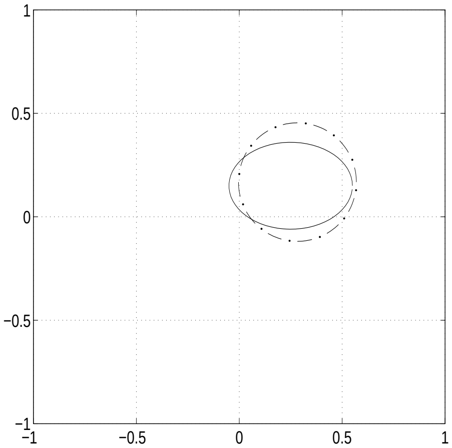
We now apply currents $g = \frac{\partial x_j}{\partial \nu}$, $j = 1, ..., d$, and compute

$$\lim_{|x|\to\infty} |x|^{d-1} H[g](x). \tag{9}$$

In this way, one can detect the first order polarization tensor $m_{\alpha\beta}$, $|\alpha| = |\beta| = 1$. Then, because of (5), we can detect the size of D. Moreover, the polarization tensors carry more information on D than the size. One way to see this is to identify an ellipse which has the same polarization tensors as detected ones. In two dimensions the polarization tensors associated with ellipses are known. See for example [10]. There is one-to-one correspondence between ellipses and two dimensional polarization tensors. Thus we can identify an ellipse which represents the detected polarization tensors. This ellipse carries information of not only the size but also some geometry of the inclusion D. To detect the center z, one can apply $g = \frac{\partial(x_1^2 - x_2^2)}{\partial\nu}$ for two dimensions and $g = \frac{\partial(x_2^2 - x_3^2)}{\partial\nu}$ as well for three dimensions, and then compute (9). We note that the same method applied to detect inclusions with anisotropic conductivities [15].

So far, we explain the method to detect single inclusion. There are not much development on the detection of multiple inclusions. The only method is to use geometric optic solution for g to develop singularities at centers [7]. In [4] plane waves were used to detect multiple inclusions entering the Helmholtz equation.

Detection of inhomogeneities inside an elastic body is also extremely important problem in material sciences and various industries. The method described in this article was applied to the system of linear elasticity [6]. We end this article by showing two numerical examples of detection of inclusions from [16].



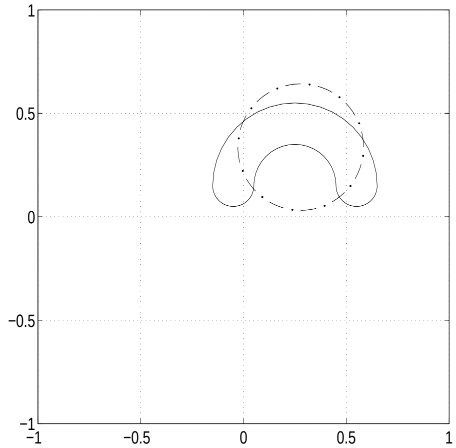


FIGURE 1. Solid lines are target inclusions and dotted lines are disks of the detected size and centers.

REFERENCES

[1] C. Alves and H. Ammari, Boundary integral formulae for the reconstruction of imperfections of small diameter in an elastic medium, SIAM J. Appl. Math. 62 (2002), 94–106.

[2] H. Ammari and H. Kang, High-Order Terms in the Asymptotic Expansions of the Steady-State Voltage Potentials in the Presence of Conductivity Inhomogeneities of Small Diameter, to appear in SIAM J. Math. Anal.

[3] ——————-, Properties of the generalized polarization tensors, to appear in SIAM J. Multiscale Modeling and Simulation.

[4] ——————, A new method for reconstructing electromagnetic inhomogeneities of small volume, to appear in Inverse Problems.

[5] ——————-, Boundary Layer Techniques for Solving the Helmholtz Equation in the Presence of Small Inhomogeneities, (2002) preprint.

[6] H. Ammari, H. Kang, G. Nakamura, and K. Tanuma, Complete Asymptotic Expansions of Solutions of the System of Elastostatics in the Presence of an Inclusion of Small Diameter and Detection of an Inclusion, to appear in J. Elasticity.

[7] H. Ammari, S. Moskow, and M. Vogelius, Boundary integral formulas for the reconstruction of electromagnetic imperfections of small diameter, to appear in ESAIM: Cont. Opt. Calc. Var.

[8] H. Ammari and J.K. Seo, A new formula for the reconstruction of conductivity inhomogeneities, to appear in Adv. in Applied Math.

[9] H. Ammari, M. Vogelius, and D. Volkov, Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of imperfections of small diameter II. The full Maxwell equations, J. Math. Pures Appl. 80 (2001), 769-814.

[10] M. Brühl, M. Hanke, and M. Vogelius, A direct impedance tomography algorithm for locating small inhomogeneities, to appear in Numer. Math.

[11] Y. Capdeboscq and M. Vogelius, A general representation formula for boundary voltage perturbations caused by internal conductivity inhomogeneities of low volume fraction, to appear *M2AN*.

[12] D.J. Cedio-Fengya, S. Moskow, and M. Vogelius, Identification of conductivity imperpections of small diameter by boundary measurements: Continuous dependence and computational reconstruction, Inverse Problems 14 (1998), 553-595.

[13] A. Friedman and M. Vogelius, Identification of small inhomogeneities of extreme conductivity by boundary measurements: a theorem on continuous dependence, Arch. Rat. Mech. Anal., 105 (1989), pp. 299–326.

[14] V. Isakov, On uniqueness of recovery of a discontinuous conductivity coefficient, Comm. Pure Appl. Math., 41 (1988), pp 865-877.
[15] H. Kang, E Kim, and K. Kim, Anisotropic polarization tensors and detection of an anisotropic inclusion, to appear in SIAM J. Appl. Math.
[16] H. Kang, E Kim, and J. Lee, Identification of elastic inclusions and elastic moment tensors, in preparation.
[17] O. Kwon, J.K. Seo, and J.R. Yoon, A real-time algorithm for the location search of discontinuous conductivities with one measurement, Comm. Pure Appl. Math. 55 (2002), 1–29.
[18] G. Pólya and G. Szegö, Isoperimetric Inequalities in Mathematical Physics, Annals of Mathematical Studies Number 27, Princeton University Press, Princeton 1951.
[19] M. Schiffer and G. Szegö, Virtual mass and polarization, Trans. AMS 67 (1949), 130-205.
[20] M. Vogelius and D. Volkov, Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities, Math. Model. Numer. Anal. 34 (2000), 723-748.

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