Trends in Mathematics
Information Center for Mathematical Sciences
Volume 3, December 2000, Pages 112–122

INTERPOLATION SERIES OF CONTINUOUS FUNCTIONS IN LOCAL FIELDS

SANGTAE JEONG

Abstract. We introduce a variety of orthonormal bases to characterize continuous functions defined on the integer rings of local fields.

1. Introduction

Since Mahler's expansion theorem for p-adic continuous functions, much progress has recently been made on the characterization of continuous functions defined over the integer rings of local fields. By introducing useful criteria to characterize such continuous functions, we illustrate several examples of orthonormal bases in spaces of continuous functions. As is well known, the importance of orthonormal bases lies in the study of the space of measures, viewed as the dual space of continuous functions.

Let K be a local field of any characteristic with the ring $\mathcal{O}$ of integers of K, and let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$ with a uniformizer π. We denote the residue field by $\mathbb{F} = \mathcal{O}/\mathfrak{m}$ of cardinality q. Then every non-zero element of K is written uniquely as a Laurent series in π with coefficients in $\mathbb{F}$. Let $|\cdot|$ be the absolute value on K associated with a normalized discrete valuation v of K. Then K is a non-Archimedean complete field with respect to $|\cdot|$. Typical examples of local fields are $\mathbb{Q}_p, \mathbb{F}_q((T))$ and finite extensions of $\mathbb{Q}_p, \mathbb{F}_q((T))$, respectively.

Let $(E, \|\cdot\|)$ be a Banach space over a local field K with a norm $\|\cdot\|$ satisfying the ultra-metric inequality: for $\quad f, g \in E\ ,\ \|f+g\| \leq \max\{\|f\|, \|g\|\}$.

For $E_0 := \{f \in E : \|f\| \leq 1\}$, we form the residual space $\overline{E} := E_0/\mathfrak{m}E_0$, which turns out to be a vector space over the residue field $\mathbb{F}$. Throughout the paper, we assume $\|E\| = |K|$, by which we mean here that every non-zero element of E has its norm value in the value group of K so that elements of E can be scaled to have norm 1. A concrete example of such a Banach space is $E := C(\mathbb{Z}_p, \mathbb{Q}_p)$, the space of p-adic continuous functions from $\mathbb{Z}_p$ to $\mathbb{Q}_p$, equipped with the sup-norm $\|f\| = \sup_{x\in\mathbb{Z}_p}\{|f(x)|\}$. It is easy then to check the residual space of E, $\overline{C(\mathbb{Z}_p, \mathbb{Q}_p)}$ is isomorphic to $C(\mathbb{Z}_p, \mathbb{F}_p)$ and we may take another example of such a Banach space over a local field of positive characteristic, say $\mathbb{F}_q((T))$. More generally, for a given local field K, we denote by $C(\mathcal{O}, K)$ the K-Banach space of continuous functions from $\mathcal{O}$ to K, topologized by the sup-norm. In this paper we aim to survey a variety of orthonormal bases to characterize elements of $C(\mathcal{O}, K)$ and give a closed

---

2000 *Mathematics Subject Classification.* 46S10,11S80,26E50.
Key words and phrases. local fields, orthonormal basis, binomial polynomials, q-analogue, Carlitz polynomials, hyperdifferential operators, digit derivatives, digit principle.
Received August 31, 2000.

formula for expansion coefficients involving the difference operators, if necessary. Indeed, this paper is based on recent work by K. Conrad [Co2], in which a general theory of orthonormal bases is developed and on the author's work in local fields of characteristic $p > 0$ [J2].

2. Criteria for orthonormal bases

To begin with, we introduce several criteria of orthonormal bases for $E := C(\mathcal{O}, K)$, which will be used to give much simpler proofs for known orthonormal bases such as Mahler's binomial coefficient polynomials and to construct orthonormal bases out of a collection of known functions via the digit expansion. Before proceeding we give a definition of what an orthonormal basis is in a Banach space E.

Definition 2.1. *Let K be a local field, and E be a K-Banach space equipped with a norm $\|\cdot\|$. We say that a sequence $\{f_n\}_{n\geq 0}$ in E is an orthonormal basis for E if and only if the following conditions are satisfied:*

(1) *every $f \in E$ can be expanded uniquely as $f = \sum_{n=0}^{\infty} a_n f_n$, with $a_n \in K$.*

(2) *$\{a_n\}$ is a null sequence in K, that is $a_n \to 0$.*

(3) *The norm of f is given by $\|f\| = max\{|a_n|\}$.*

The first criterion of orthonormal bases is due to Serre [Se].

OBC 1. *Let K be a local field and E be a K-Banach space so that $\|E\| = |K|$. A sequence $\{f_n\} \in E$ is an orthonormal basis of E if and only if*

(1) *every function f_n lies in E_0.*

(2) *the reduced functions $\overline{f_n} \in \overline{E}$ form an $\mathbb{F}$-basis of $\overline{E}$ in the algebraic sense, where $\mathbb{F} = \mathcal{O}/\mathfrak{m}$ is the residue field.*

Since $C(\mathcal{O}, \mathbb{F}) = \lim_{\rightarrow} \mathrm{Maps}(\mathcal{O}/\mathfrak{m}^n, \mathbb{F})$, **OBC 1** is reduced to a linear algebra problem: A necessary and sufficient condition for $\{f_n\}_{n\geq 0}$ to be an orthonormal basis for $C(\mathcal{O}, K)$ is that

(a) each f_n maps $\mathcal{O}$ into itself.

(b) for any $n \geq 0$, the reduced functions $\overline{f}_0, \cdots \overline{f}_{q^n-1}$, are linearly independent over $\mathbb{F}$ as elements of $\mathrm{Maps}(\mathcal{O}/\mathfrak{m}^n, \mathbb{F})$.

¿From now on, the functions satisfying part (a) are said to be *integral-valued*. We now give a slight modified version of Serre's criterion, which is useful to use only if there is a known orthonormal basis to compare with.

OBC 2. *Let K be a local field and E be a K-Banach space with an orthonormal basis $\{e_n\}_{n\geq 0}$. If $f_n \in E$ with $\sup_{n\geq 0} \|f_n - e_n\| < 1$ for every n, then $\{f_n\}_{n\geq 0}$ is an orthonormal basis of E.*

OBC 2 follows at once from **OBC 1** but we refer to [Co1] for a direct proof using the norm inequality. Recently K. Conrad used both **OBC 1** and linear algebra to give a construction of orthonormal bases out of known functions via q-adic digit expansion, which is called *digit principle*(**OBC 3** and **OBC 4**).

Definition 2.2. (1) *Write a non-negative integer j in base q as*

$$j = \alpha_0 + \alpha_1 q + \cdots + \alpha_{n-1} q^{n-1} \text{ with } 0 \leq \alpha_k < q,$$

Then the sum of digits of j is defined as $s(j) = \alpha_0 + \alpha_1 + \cdots + \alpha_{n-1}$.

(2) *For a given family of functions $\{e_n\}_{n\geq 0}$, the sequence $\{f_j\}_{j\geq 0}$ is called the extension of the sequence e_n by q-digits if for each non-negative integer j,*

$$f_j := {e_0}^{\alpha_0} {e_1}^{\alpha_1} \cdots e_{\alpha_{n-1}}^{\alpha_{n-1}}.$$

OBC 3. *Let K be a local field and let H_n be a sequence of open subgroups of $\mathcal{O}$ such that $H_{n+1} \subset H_n$ and $\cap H_n = 0$. Suppose there is a sequence of integral-valued functions, $\{e_n\} \in C(\mathcal{O}, K)$ and for all $n \geq 1$, the reductions $\overline{e}_0, \cdots, \overline{e}_{n-1} \in C(\mathcal{O}, \mathbb{F})$ are constants on the cosets of H_n and the map*

$$\mathcal{O}/H_n \to \mathbb{F}^n, x \mapsto (\overline{e}_0, \cdots, \overline{e}_{n-1}(x))$$

is bijective. Then the extension of the sequence e_n by q-digits gives an orthonormal basis of $C(\mathcal{O}, K)$.

In local fields K of positive characteristic, **OBC 3** can be much simplified to the following because, unlike p-adic cases, $C(\mathcal{O}, K)$ has the closed subspace consisting of $\mathbb{F}$-linear continuous functions from $\mathcal{O}$ to K, which we denote by $LC(\mathcal{O}, K)$.

OBC 4. *If $\{e_n\}_{n\geq 0}$ is an orthonormal basis of $LC(\mathcal{O}, K)$, then the extension of the sequence e_n by q-digits is an orthonormal basis of $C(O, K)$.*

Tateyama [Ta] obtained a criterion that a sequence of polynomials $\{f_j\}_{j\geq 0}$ being a orthonormal basis of $C(\mathcal{O}, K)$ can be given in terms of degrees and leading coefficients (lc for short). To state it, we denote by $P(\mathcal{O}, \mathcal{O})$ the $\mathcal{O}$-module generated by integral-valued polynomials, which is known dense in $C(\mathcal{O}, K)$.

OBC 5. *If a sequence of polynomials f_j of degree $j \geq 0$ in $P(\mathcal{O}, \mathcal{O})$ satisfies*

$$v(lc(f_j)) = -\frac{j - s(j)}{q - 1},$$

then the sequence $\{f_j\}_{j\geq 0}$ forms an orthonormal basis of $C(O, K)$.

At last we close this section by giving a (density) criterion known in non-Archimedean analysis. In order to use it we need an orthogonality condition for elements in a Banach space [S].

OBC 6. *Let K be a non-Archimedean complete local field, and E be a K-Banach space, and $\{f_j\}_{j\geq 0}$ be a sequence of orthonormal elements whose $Span_K\{f_j : j \geq 0\}$ is dense in E. Then $\{f_j\}_{j\geq 0}$ is an orthonormal basis for E.*

3. Orthonormal Bases

3.1. **Characteristic 0.** In 1944 Dieudonné [Di] proved an analogue of the Weierstrass approximation theorem for continuous functions defined on compact subgroups of local fields. In 1958 Mahler [M] sharpened this result by showing that any continuous function f on $\mathbb{Z}_p$ is the uniform limit of an interpolation series $f = \sum_{n=0}^{\infty} a_n \binom{x}{n}$, where the coefficients a_n are uniquely recovered by iterating the difference operators Δ:

$$a_n = (\Delta^n f)(0) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(n-k),$$

where $\Delta f(x) = f(x+1) - f(x)$. In a simple terms described in Definition 2.1, we can restate Mahler' result in terms of the binomial coefficient polynomials .

Theorem 3.1. *$\{\binom{x}{n}\}_{n\geq 0}$ is an orthonormal basis of $C(\mathbb{Z}_p, \mathbb{Q}_p)$. Moreover the coefficients a_n are uniquely recovered by the difference operator given above.*

A simple proof of this result is given using a reduced version of **OBC 1**. Since

$$a \equiv b \bmod p^n \Rightarrow \binom{a}{i} \equiv \binom{b}{i}$$

for $0 \leq i \leq p^n - 1$, so $\binom{x}{i}$ make sense in $\mathrm{Maps}(\mathbb{Z}/p^n\mathbb{Z}, \mathbb{F}_p)$. Thus it suffices to check that these functions span $\mathrm{Maps}(\mathbb{Z}/p^n\mathbb{Z}, \mathbb{F}_p)$ as an $\mathbb{F}_p$-vector space. In fact, this can be done by showing that the spanning set of characteristic functions at all points of $\mathbb{Z}/p^n\mathbb{Z}$ is in the span space of $\{\binom{x}{i}\}_{0\leq i\leq p^n-1}$.

Since Mahler, different proofs of his theorem were given by Amice [Am] and Bojanic [Bo]. In particular, Amice generalized Mahler's theorem to any finite extension of $\mathbb{Q}_p$. Let K be a local field, $\mathcal{O}$ the ring of integers, $\mathfrak{m} = (\pi)$ the maximal ideal of $\mathcal{O}$, $\mathbb{F} = \mathcal{O}/\mathfrak{m}$ the residue class field of q elements and $S = \{\beta_0, \beta_1, \cdots, \beta_{q-1}\}$ be a system of representatives of $\mathbb{F}$. For an integer $n \geq 0$, written q-adically by

$$n = \alpha_0 + \alpha_1 q + \cdots + \alpha_s q^s \text{ with } 0 \leq \alpha_k < q,$$

we associate the element

$$u_n = \beta_{\alpha_0} + \beta_{\alpha_1}\pi + \cdots + \beta_{\alpha_s}\pi^s.$$

With these elements we form a sequence of Newton-type interpolation polynomials as follows:

$$P_0(x) = 1, P_n(x) = (x-u_0)(x-u_1)\cdots(x-u_{n-1}),$$
$$Q_0(x) = 1, Q_n(x) = P_n(x)/P_n(u_n).$$

Theorem 3.2. *$\{Q_n(x)\}_{n\geq 0}$ is an orthonormal basis of $C(\mathcal{O}, K)$. In particular, when $K = \mathbb{Q}_p, \mathcal{O} = \mathbb{Z}_p, \pi = p, S = \{0, 1, , \cdots, p-1\}$, then $Q_n(x) = \binom{x}{n}$.*

Caenepeel [Cae] also gave a simple generalization of Mahler's theorem by using a reduced version of **OBC 1**, which states

Theorem 3.3. *For each non-negative integer m, $\{\binom{x}{n}^m\}_{n\geq 0}$ is an orthonormal basis of $C(\mathbb{Z}_p, \mathbb{Q}_p)$.*

We here introduce a q-analogue of Mahler's expansion theorem, which is developed by K. Conrad [Co1]. For now, let K be a finite extension of $\mathbb{Q}_p$ and $q \in K$ with $|q-1| < 1$, then we may not confuse q in K with another q denoting the cardinality of the residue field $\mathbb{F}$, because the distinction will be clear from the contexts. For an integer n and an indeterminate q, we define the q-analogue of n:

$$(n)_q = \frac{q^n - 1}{n - 1},$$

and then the q-factorial is defined using $(n)_q$ in such a similiar way as the classical fatorial. The q-binomial coefficient for non-negative integers m and n with $m \geq n$ is defined as

$$\binom{m}{n}_q = \frac{(m)_q!}{(n)_q!(m-n)_q!}.$$

Note that $\binom{m}{n}_1 = \binom{m}{n}$. We then formulate a q-analogue of Mahler's theorem.

Theorem 3.4. *Let K be a finite extension of $\mathbb{Q}_p$ and $q \in K$ with $|q-1| < 1$.*
(1) $\{\binom{x}{n}_q\}_{n\geq 0}$ *is an orthonormal basis of $C(\mathcal{O}, K)$.*
(2) *Moreover, writing $f = \sum_{n\geq 0} c_{n,q}\binom{x}{n}_q$, the coefficients are given by*

$$c_{n,q} = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{(n-k)(n-k-1)/2} f(k).$$

Four different proofs of Theorem 3.4 are known but we here give a simple proof among them, which depends on Mahler's theorem and an inequality involving q-binomial coefficients [Co1]. This inequality implies that

$$|\binom{x}{n}_q - \binom{x}{n}| \leq |q-1| < 1,$$

so we are done by **OBC 2**.

We now turn to some applications of the digit principle.

Theorem 3.5. *For $n \geq 0$, write $n = \alpha_0 + \alpha_1 p + \cdots + \alpha_s p^s$ with $0 \leq \alpha_k < p$. Set*

$$\{\frac{x}{n}\} := \binom{x}{1}^{\alpha_0}\binom{x}{p}^{\alpha_1}\cdots\binom{x}{p^s}^{\alpha_s}.$$

Then $\{\{\frac{x}{n}\}\}_{n\geq 0}$ is an orthonormal basis of $C(\mathbb{Z}_p, \mathbb{Q}_p)$.

To prove Thoerem 3.5, it suffices to show by **OBC 4** that for each $x \in \mathbb{Z}_p/p^n\mathbb{Z}_p$, the functions

$$\binom{x}{1} \bmod p, \binom{x}{p} \bmod p, \cdots \binom{x}{p^{n-1}} \bmod p$$

determines x. It is easy to see the claim follows from Lucas' congruence theorem.

As an immediate consequence of Theorem 3.5 we deduce Mahler's theorem by checking that the transition matrix from $\{\frac{x}{0}\}, \cdots \{\frac{x}{n}\}$ to $\binom{x}{0}\cdots\binom{x}{n}$ is triangular with all diagonal entries having p-adic units.

Three generalizations of Mahler's Theorem are also given by van Hamme, who did not use any criteria provided in section 2 but hypergeometric series and the convolution of two sequences. We refer the reader to [Ham] for more details.

3.2. **Characteristic p.** We will characterize continuous functions in local fields of positive characteristic. Before proceeding, we first introduce some notations and analogies with p-adic fields.

Let $\mathbb{F}_q[T]$ be the ring of polynomials in one variable T over a finite field $\mathbb{F}_q$ of q elements and let $\mathbb{F}_q((T))$ be the field of formal Laurent series in T with coefficients in $\mathbb{F}_q$ with the absolute value $|\cdot|$ associated to a usual (normalized) discrete valuation v. Then the discrete valuation ring of v consists of formal power series in T, which we denote by $\mathbb{F}_q[[T]]$. We then have the well known analogies:

$$\mathbb{Z} \leftrightarrow \mathbb{F}_q[T], \mathbb{Z}_p \leftrightarrow \mathbb{F}_q[[T]], \mathbb{Q}_p \leftrightarrow \mathbb{F}_q((T)).$$

Let $C(\mathbb{F}_q[[T]], \mathbb{F}_q((T)))$ be the $\mathbb{F}_q((T))$-Banach space of continuous functions $f : \mathbb{F}_q[[T]] \to \mathbb{F}_q((T))$ with the sup-norm $\|f\| = \max_{x\in\mathbb{F}_q[[T]]}\{|f(x)|\}$ and let

$LC(\mathbb{F}_q[[T]], \mathbb{F}_q((T)))$ be the closed subspace of continuous $\mathbb{F}_q$-linear functions $f : \mathbb{F}_q[[T]] \to \mathbb{F}_q((T))$ under the metric induced from $C(\mathbb{F}_q[[T]], \mathbb{F}_q((T)))$.

Let

$$e_n(x) = \prod_{\substack{\alpha \in \mathbb{F}_q[T] \\ \deg\alpha < n}} (x - \alpha), n \geq 1; e_0(x) = x,$$

Let $F_0 = 1$ and for $n \geq 1$ let

$$F_n = [n][n-1]^q \cdots [1]^{q^{n-1}},$$

where $[n] = T^{q^n} - T$. Then it is known that F_n is the product of all monic polynomials in $\mathbb{F}_q[T]$ of degree n.

Definition 3.6. (1) *Put $E_n(x) = e_n(x)/F_n$ for any integer $n > 0$ and $E_0(x) = x$. We call $\{E_n\}_{n\geq 0}$ Carlitz linear polynomials.*

(2) *For a non-negative integer j, written in base q by*

$$j = \alpha_0 + \alpha_1 q + \cdots + \alpha_s q^s$$

with $0 \leq \alpha_i < q$, put

$$G_j(x) := \prod_{n=0}^{s} E_n^{\alpha_n}(x), j \geq 1; G_0(x) = 1,$$

and

$$G_j^*(x) := \prod_{n=0}^{s} G^*_{\alpha_n q^n}(x),$$

where

$$G^*_{\alpha q^n}(x) = \begin{cases} E_n^{\alpha}(x) & if 0 \leq \alpha < q-1 \\ E_n^{\alpha}(x) - 1 & if \alpha = q-1. \end{cases}$$

We call $\{G_j\}_{j\geq 0}$ Carlitz polynomials.

Note that these $G_j(x)$ and $G_j^*(x)$ are polynomials of degree j and they have leading coefficients $\prod_{n=0}^{s} F_n^{\alpha_n}$ which is known as an analogue of the classical factorial. Moreover, it is known in [C2] that $G_j(x)$ and $G_j^*(x)$ are integral-valued polynomials satisfying the binomial identity as the binomial polynomials$\binom{x}{n}$ do in p-adic integers .

It is necessary to recall the Carlitz difference operators [C1] to get a formula for expansion coefficients.

Definition 3.7. *Carlitz difference operators, $\Delta^{(n)}$ are defined recursively as follows:*

$$(\Delta^{(n)} f)(x) = \Delta^{(n-1)} f(Tx) - T^{q^{n-1}} \Delta^{(n-1)} f(x), n \geq 1; \Delta^{(0)} = id.$$

Following Amice's theory [Am] in p-adic fields, in 1971 C. Wagner [W1] first employed Newton type interpolation polynomials to establish a characteristic p analogue of Mahler's theorem and then another proof was given by D. Goss [Go1].

Theorem 3.8. (1) *$\{G_j(x)\}_{j\geq 0}$ is an orthonormal basis of $C(\mathbb{F}_q[[T]], \mathbb{F}_q((T)))$.*

(2) *Write $f = \sum_{j\geq 0} A_j G_j(x)$. Then A_j can be determined by the formula: For any integer n such that $q^n > j$,*

$$A_j = (-1)^n \sum_{\substack{\alpha \in \mathbb{F}_q[T] \\ deg(\alpha) < n}} G^*_{q^n-1-j}(\alpha) f(\alpha).$$

Wagner [W3] also gave an alternate proof of Theorem 3.8 by directly showing that the series $\sum_{n=0}^{\infty} A_j G_j(x)$ converges uniformly to $f(x)$ on $\mathbb{F}_q[[T]]$ where A_j is given above. As a corollary to Theorem 3.8 he showed that

Corollary 3.9. *Let f be given as $f = \sum_{j \geq 0} A_j G_j(x)$. Then $f \in LC(\mathbb{F}_q[[T]], \mathbb{F}_q((T)))$ if and only if $A_j = 0$ for $j \neq q^n$, where $n \geq 0$.*

Wagner [W2] used non-Archimedean analysis to give another proof of Corollary 3.9 by showing for a given continuous $\mathbb{F}_q$-linear function f that the series $\sum_{n=0}^{\infty} A_{q^n} E_n(x)$ converges to uniformly to $f(x)$ on $\mathbb{F}_q[[T]]$.

Theorem 3.10. (1) *$\{E_n(x)\}_{n \geq 0}$ is an orthonormal basis of $LC(\mathbb{F}_q[[T]], \mathbb{F}_q((T)))$.*

(2) *Write $f = \sum_{n=0}^{\infty} a_n E_n(x)$. The coefficients $\{a_n\}_{n \geq 0}$ can then be recovered by the Carlitz difference operators $\Delta^{(n)}$:*

$$a_n = \Delta^{(n)} f(1).$$

In the reverse way to Wagner's, we use the digit principle(**OBC 4**) to give much simpler proofs of the preceding two theorems. To prove Theorem 3.10, by applying (reduced) **OBC 1** to $LC(\mathbb{F}_q[[T]], \mathbb{F}_q((T)))$ we need only to check that for any $n \geq 1, \overline{E}_0, \overline{E}_1 \cdots \overline{E}_{n-1}$ are linearly independent over $\mathbb{F}_q$ as elements of Maps$(\mathbb{F}_q[T]/T^n, \mathbb{F}_q)$. Suppose that

$$\alpha_0 \overline{E}_0(x) + \alpha_1 \overline{E}_1(x) + \cdots + \alpha_{n-1} \overline{E}_{n-1}(x) = 0.$$

Plugging $1, T, \cdots T^{n-1}$ into x successively gives us $\alpha_i = 0$ for $0 \leq i \leq n-1$, as $E_i(T^i) = 1$ for all i. Therefore we get Theorem 3.10 and the digit principle implies Theorem 3.8 . The closed formula for expansion coefficients follows from Carlitz's integral formula [C2], which is based on the (binomial) identities Carlitz polynomials G_j and G_j^* satisfy. In linear cases, the coefficient formula is rather easy to derive.

We mention here that a variant of Theorem 3.8 follows from computing the valuation of leading coefficients of Carlitz polynomials $G_j^*(x)$ [Go2] together with **OBC 5**.

Theorem 3.11. *$\{G_j^*(x)\}_{j \geq 0}$ is an orthonormal basis of $C(\mathbb{F}_q[[T]], \mathbb{F}_q((T)))$.*

There is another class of $\mathbb{F}_q$-linear non-polynomial functions which have a close similiarity with Carlitz linear polynomials. We introduce a sequence of $\mathbb{F}_q$-linear operators, called hyperdifferential operators, $\mathcal{D}_n, n \geq 0$, defined by

$$\mathcal{D}_n(\sum a_i T^i) = \sum \binom{i}{n} a_i T^{i-n}.$$

These operators were first introduced by Hasse [Has] and then studied by Hasse and Schmidt [HS] and Teichmuller [Tei]. Moreover, it is known that they are $\mathbb{F}_q$-linear continuous operators defined over $\mathbb{F}_q[[T]]$ and like the ordinary higher derivatives, hyperdifferential operators satisfy various properties such as the general product rule, quotient rule and chain rule, etc.

Definition 3.12. (1) *For a non-negative integer* j, *written in base* q *by*

$$j = \alpha_0 + \alpha_1 q + \cdots + \alpha_s q^s$$

with $0 \le \alpha_i < q$, *put*

$$\mathsf{D}_j(x) := \prod_{n=0}^{s} \mathcal{D}_n^{\alpha_n}(x), j \ge 1; \mathsf{D}_0(x) = 1.$$

(2) *Set*

$$\mathsf{D}_j^*(x) := \prod_{n=0}^{s} \mathsf{D}^*_{\alpha_n q^n}(x),$$

where

$$\mathsf{D}^*_{\alpha q^n}(x) = \begin{cases} \mathcal{D}_n^{\alpha}(x) & if 0 \le \alpha < q-1 \\ \mathcal{D}_n^{\alpha}(x) - 1 & if \alpha = q-1. \end{cases}$$

Parallel to Wagner's result(Theorem 3.10), we get the following

Theorem 3.13. (1) $\{\mathcal{D}_n(x)\}_{n\ge 0}$ *is an orthonormal basis for* $LC(\mathbb{F}_q[[T]], \mathbb{F}_q((T)))$.
(2) *Write* $f = \sum_{n=0}^{\infty} b_n \mathcal{D}_n(x)$. *The coefficients can be recovered by iterating the Carlitz difference operator* Δ:

$$b_n = (\Delta^n f)(1) = \sum_{i=0}^{n} (-1)^{n-i} f(T^i)\mathcal{D}_i(T^n),$$

where $\Delta := \Delta^{(1)}$ *is defined above.*

Three different proofs of Theorem 3.13 are given by Snyder [Sn], the author [J1] and Conrad [Co2]. We mention that the first two proofs depend on Wagner's result. In fact Snyder establishes the result by imitating Wagner's arguments on non-Archimedean analysis and the author's proof is based on **OBC 6**. On the other hand, Conrad uses **OBC 4** to give a simple and alternate proof, which is independent of Wagner's theorem. Specifically as in Carlitz linear polynomials, we are done by showing that as $\mathbb{F}_q$-linear functionals of $\mathbb{F}_q[T]/T^n$, the reduced functions, $\overline{\mathcal{D}}_0, \overline{\mathcal{D}}_1 \cdots, \overline{\mathcal{D}}_{n-1}$ are dual to a basis $1, T, \cdots, T^{n-1}$ for the vector space $\mathbb{F}_q[T]/T^n$.

The coefficient formula is immediate from the observation that Δ acts on $\mathcal{D}_n$ by shifting, namely $\Delta\mathcal{D}_n = \mathcal{D}_{n-1}$. But the explicit formula for $(\Delta^n f)(1)$ is given by induction n. Passing to q-adic extension of hyperdifferential operators, the first part of Theorem 3.14 is immediate from **OBC 6**. As for the coefficient formula it can be obtained in two different ways. See [J2],[J3] for more details on this.

Theorem 3.14. (1)$\{\mathsf{D}_j(x)\}_{j\ge 0}$ *are orthonormal bases of* $C(\mathbb{F}_q[[T]], \mathbb{F}_q((T)))$.
(2) *Write* $f \in C(\mathbb{F}_q[[T]], \mathbb{F}_q((T)))$ *as* $f = \sum_{j\ge 0} B_j \mathsf{D}_j(x)$. *Then* B_j *can be recovered by the formula: For any integer* n *such that* $q^n > j$,

$$B_j = (-1)^n \sum_{\substack{\alpha \in \mathbb{F}_q[T] \\ deg(\alpha) < n}} \mathsf{D}^*_{q^n-1-j}(\alpha) f(\alpha).$$

A close relation between Carlitz linear polynomials and hyper-differential operators is observed by the author's using **OBC 2** [J2]. The two bases are essentially equivalent, which means here that Carlitz linear polynomials being an orthonormal basis for $LC(\mathbb{F}_q[[T]], \mathbb{F}_q((T)))$ can be deduced from knowing that hyper-differential operators are an orthonormal basis and vice versa. The equivalence relation between the two bases of the subspace $LC(\mathbb{F}_q[[T]], \mathbb{F}_q((T)))$ gives us that both Carlitz polynomials $\{G_j\}_{j\geq 0}$ and digit derivatives $\{\mathsf{D}_j\}_{j\geq 0}$ have the same images in the reduced space $C(\mathbb{F}_q[[T]], \mathbb{F}_q)$. Thus we see with no further work that the q-adic extensions of these two bases in $LC(\mathbb{F}_q[[T]], \mathbb{F}_q((T)))$ are orthonormal bases of $C(\mathbb{F}_q[[T]], \mathbb{F}_q((T)))$. We finally give a characteristic p analogue of Theorem 3.3, which follows again from **OBC 2**.

Theorem 3.15. *Let X be a compact subset of $\mathbb{F}_q((T))$, or more generally any compact Hausdorff totally disconnected topological space and let $\{f_j\}_{j\geq 0}$ is an orthonormal basis of $C(X, \mathbb{F}_q((T)))$. Then $\{f_j^{q^m}\}_{j\geq 0}$ is an orthonormal basis for $C(X, \mathbb{F}_q((T)))$ for all positive integer m.*

3.3. **Any characteristic.** Let K be a local field of any characteristic, $\mathcal{O}$ the ring of integers of K, $\mathfrak{m}$ a uniformizer of $\mathcal{O}$, $\mathbb{F} = \mathcal{O}/\mathfrak{m}$ the residue field of $\mathcal{O}$ of cardinality q. We first define the so-called Fermat quotient polynomial δ given by

$$\delta(x) = \frac{x - x^q}{\pi}$$

and its iterates $\delta^n, n \geq 1$ with $\delta_\pi^0(x) = x$. Then we see easily that each δ^n is an integral-valued polynomial of degree q^n. For a non-negative integer j with

$$j = \alpha_0 + \alpha_1 q + \cdots + \alpha_s q^s,$$

where $0 \leq \alpha_i < q$, we associate the function

$$\Phi_j(x) := (\delta^0 x)^{\alpha_0} (\delta^1 x)^{\alpha_1} \cdots (\delta^s x)^{\alpha_s}, j \geq 1; \Phi_0(x) := 1.$$

Then Φ_j is also an integral-valued polynomial of degree j and has leading coefficient of valuation $-\frac{j-s(j)}{q-1}$. Thus the following is immediate from **OBC 5**.

Theorem 3.16. $\{\Phi_j(x)\}_{j\geq 0}$ *is an orthonormal basis of* $C(\mathcal{O}, K)$.

This result can be also derived from **OBC 4** by showing that the map $\mathcal{O}/\pi^n \to {\mathbb{F}_q}^n$ defined by $x \mapsto (\overline{\delta}_0(x), \cdots, \overline{\delta}_{n-1}(x))$ is bijective. Furthermore, by forming a small variant of Fermat quotient polynomials Buium [Bui] generalizes Theorem 3.16 to the case where K is a Galois extension of a finite extension k of $\mathbb{Q}_p$ and the residue field of k is of cardinality q. We also refer the reader to [CC] for another orthonormal basis for $C(\mathcal{O}, K)$ consisting of the iterates of the Fermat quotient polynomials in the case where $\mathcal{O}$ is unramified over $\mathbb{Z}_p$.

The construction in Theorem 3.16 is formulated more generally by Tateyama [Ta] using coefficient functions arising from Lubin-Tate formal groups. Conrad [Co2] develops this method systematically using the digit principle. Fix a uniformizer π and a Lubin-Tate formal group $F/\mathcal{O}$ associated to some Frobenius power series $[\pi](X) \in \mathcal{O}[[X]]$. For an endomorphism of F attached to $a \in \mathcal{O}$, write

$$[a](X) = [a]_F(X) + \sum_{n\geq 1} C_{n,F}(a) X^n.$$

Then using the formal group law and this equation gives us that the coefficient functions $C_{n,F}$ are integral-valued polynomials of degree at most n in any characteristic. We give two examples of this case, one in each characteristic. Let $F/\mathbb{Z}_2$ be the Lubin -Tate formal group attached to $[2](X) = X^2 + 2X = (1+X)^2 - 1$, Then we see F is the multiplicative group and $C_{n,F} = \binom{x}{n}$. The second example is the

Carlitz module given by the Lubin-Tate group over $\mathbb{F}_q[[T]]$ attached to the series $[T](X) = X^q + TX$. Then

$$C_{n,F}(a) = \begin{cases} E_k(a) & if n = q^k, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.17. *Let $\mathcal{O}$ be the integer ring of a local field, $\mathbb{F}$ the residue field of cardinality q. For a Lubin-Tate group $F/\mathcal{O}$, the polynomials*

$$C_{1,F}{}^{\alpha_0} C_{q,F}{}^{\alpha_1} \cdots C_{q^n,F}{}^{\alpha_n}, n \geq 0, 0 \leq \alpha_j \leq q-1$$

is an orthonormal basis of $C(\mathcal{O}, K)$.

Our last example is an orthonormal basis due to Baker [Ba], consisting not of polynomials, but of locally constant functions taking values in the Teichmüller representatives. This also follows from **OBC 3**.

Theorem 3.18. *Let K be any local field, $\mathcal{O}$ its ring of integers, π a fixed uniformizer of $\mathcal{O}$, $q = \#\mathcal{O}/\pi$. Fro each $x \in \mathcal{O}$, write*

$$x = \sum_{i \geq 0} \omega_i(x) \pi^i,$$

where $\omega_i(x)$ is a Teichmüller representative. Then the q-adic extension of $\omega_i(x)$, $\Omega_j(x)$ is an orthonormal basis of $C(\mathcal{O}, K)$, where

$$\Omega_j(x) := \omega_0(x)^{\alpha_0} \omega_1(x)^{\alpha_1} \cdots \omega_s(x)^{\alpha_s}$$

Finally one may think of many applications of orthonormal bases to different settings, for example, the space of measures. But here we only mention the characterization of analytic or locally analytic functions in terms of the valuations of expansion coefficients with respect to the binomial coefficient polynomials or Carlitz polynomials. For a detailed reference, see [Am] and [Y].

Acknowledgement.

This paper was partially supported by BK21 Project at Korea University

References

[Am] Y. Amice, Interpolation p-adique, *Bull. Soc. Math. France* **92**(1964), 117-180.

[Ba] A. Baker, p-adic continuous functions on rings of integers and a theorem of K. Mahler, *J. London Math. Soc.* **33**(1986), 414-420.

[Bo] R. Bojanic, A simple Proof of Mahler's Theorem on Approximation of Continuous Functions of p-adic Variable by Polynomials, *J. of Number Theory* **6**(1974), 412-415.

[Bui] A. Buium, Continuous π-adic Functions and π-Derivations, *J. of Number Theory* **84**(2000), 34-39.

[C1] L. Carlitz, On certain functions connected with polynomials in a Galois field, *Duke Math. J.* **1**(1935), 137-168.

[C2] L. Carlitz, A set of polynomials, *Duke Math. J.* **6**(1940), 486-504.

[Cae] S. Caenpeel, About p-adic interpolation of continuous and differential operators, *Groupe d'étude d'Analyse untramétrique* (Y. Amice, G. Christol, P. Robba) 9e année 1981/1982.$n°$ 25.

[CC] P.J. Cahen and J.-L. Chabert, "Integer Valued Polynomials," Mathematical Surveys and Monographs, **48** *Amer. Math. Soc.*, Providence, RI, 1997.

[Co1] K. Conrad, A q-Analogue of Mahler Expansions I, to appear in *Advances in Mathematics.*

[Co2] K. Conrad, The Digit Principle, *J. of Number Theory* **84**(2000), 230-257.

[Di] J. Dieudonné, Sur les fonctions continues p-adiques, *Bull. Sci. Math.* **68**(1944), 79-95.

[Go1] D. Goss, Fourier series, Measures and Divided Power Series in the theory of Function Fields, *K-theory* **1**(1989), 533-555.

[Go2] D. Goss, " Basic Structures of Function Field Arithmetic", Springer, Berlin, 1996.

[Ham] L. V. Hamme, "Continuous operators which commute with translations, on the space of continuous functions on $\mathbb{Z}_p$", p-adic functional analysis (edited by J. M. Bayod, N. Degrande-Dekimpe, J. Martinez-Maurica).

[Has] H. Hasse, Theorie der höheren Differentiale in einem algebraischen Funktionenkörper mit Vollkommenem Konstantenkörper bei beliebiger Charaicterstik, *J. Reine angew. Math.* 175 (1936), 50-54.

[HS] H. Hasse and F. K. Schmidt, Noch eine Begründung der Theorie der höheren Dieeerentialquotienten in einem algebraischen funktionenkoörper einer Unbestimmten, Die Wronskische Determinante in beliebigen differenzierbaren, *J. Jeine Angew. Math.* **177**(1937), 215-237.

[J1] S. Jeong, "Diophantine Problems in Function Fields of Positive Characteristic," Ph. D. Thesis, University of Texas at Austin, (1999).

[J2] S. Jeong, A Comparison of the Carlitz and Digit Derivatives Bases in Function Field Arithmetic, *J. of Number Theory* **84**(2000), 258-275.

[J3] S. Jeong, Hyperdifferential operators and continuous functions on function fields, to appear in *J. of Number Theory.*

[M] K. Mahler, An interpolation series for a continuous function of a p-adic variable, *J. Jeine Angew. Math.* **199** (1958), 533-555.

[S] W. Schikhof, "Ultrametric Calculus," Cambridge University Press, Cambridge, UK, 1984.

[Sch] F. K. Schmidt, Die Wronskische Determinante in beliebigen differenzierbaren Funktionenkörpern, *Math. Z.* **45** (1939), 62-74.

[Se] J. P. Serre, Endomorphismes complétement continus des espaces de Banach p-adiques, *Inst. Hautes Études Sci. Publ. Math.* **12** (1962), 69-85.

[Sn] B. Snyder, "Hyperdifferential Operators on Function Fields and Their Applications," Ph. D. Thesis, Ohio State University(Coulmbus)(1999)

[Ta] K. Tateyama, Continuous functions on discrete valuation rings, *J. of Number Theory* **75**(1999), 23-33.

[Tei] O. Teichmüller, Differentialrechung bei Charakteristik p, *J. Reine Angew. Math.* **175**(1936), 89-99.

[W1] C. G. Wagner, Interpolation series for continuous functions on π-adic completions of $\mathrm{GF}(q,x)$, *Arta Arith.* **17**(1971), 389-406.

[W2] C. G. Wagner, Linear operators in local fields of prime characteristic, *J. Reine Angew. Math.* **251**(1971), 153-160.

[W3] C. G. Wagner, Interpolation series in local fields of prime characteristic, *Duke. Math. J.* **39**(1972), 203-210.

[Y] Z. Yang, "Non-Archimedean Analysis Over Function Fields with positive characteristic," Ph. D. Thesis, Ohio State University(Coulmbus)(1999)

Department of Mathematics, Korea University, Seoul, Korea 136-701
E-mail address: stj@semi.korea.ac.kr