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ON THE CLASS NUMBER ONE PROBLEM FOR NORMAL CM-FIELDS

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ABSTRACT. In this paper, we will survey some of the recent works on the class number one problem for normal CM-fields.

1. INTRODUCTION

In 1870, Gauss has conjectured the following three problems [6]:

1. There are exactly 9 imaginary quadratic fields $\mathbb{Q}(\sqrt{d})$ of class number one, that is $d = -3, -4, -7, -8, -11, -19, -43, -67, -163$.
2. For a given integer $h > 0$, there exist only finitely many imaginary quadratic fields of class number h.
3. There exist infinitely many real quadratic fields of class number one.

Conjecture 2 follows easily from Siegel's Theorem ([29], 1935) and Conjecture 1 was proven by Stark ([30], 1967). But Conjecture 3 is still open. It is of interest to solve the class number one problem of arbitrary algebraic number fields. But since this problem is not an easy task in general, we are interested in that of normal CM-fields. It is well known that, due to A.M. Odlyzko ([24], 1975), there are only finitely many normal CM-fields with class number one and J. Hoffstein ([8], 1979) showed that their degrees are less than 436. The determination of all such fields with a given class number and given degree stems from lower bounds for the relative class number. In 1994, S. Louboutin established lower bounds for relative class number which enable to us give reasonable upper bounds for its discriminant with small class number (see [11]). For solving this problem, there are two different approaches depending on whether we fix the Galois group $\mathrm{Gal}(N/\mathbb{Q})$ of normal CM-field N (for example, abelian or dihedral) or for a given degree $2n$, we look for all the possible Galois groups of order $2n$.

Up to now, the class number one problem for imaginary abelian number fields, dihedral CM-fields, dicyclic CM-fields or the normal CM-fields of degree less than 48 excluding those of degree 32 has completely solved by many authors (see [33], [18], [19], [9], [16], [10], [25], [3], [22]). Recently, that of degree 32 and 48 is partially solved (see [34]) and [4]).

In this paper, we will survey the different techniques used to solve the class number one problem for normal CM-fields of a given Galois groups or of degree less than or equal to 48 excluding those of degree of 32.

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2. Prerequisites

Throughout this paper, for any number field k, let $\mathcal{O}_k$ denote its ring of integers, d_k the absolute value of its discriminant, h_k its class number and ζ_k its Dedekind zeta function. Recall that a CM-field is a totally imaginary number field N which is a quadratic extension of its maximal totally real subfield N^+. In this situation, the class number h_{N^+} of N^+ divides the class number h_N of N and $h_N^- = h_N/h_{N^+}$ is called the relative class number of N. Set $[N:\mathbb{Q}] = 2n$. We have

$$h_N^- = \frac{Q_N w_N}{(2\pi)^n}\sqrt{\frac{d_N}{d_{N^+}}}\frac{\mathrm{Res}_{s=1}(\zeta_N)}{\mathrm{Res}_{s=1}(\zeta_{N^+})} \tag{1}$$

where $Q_N \in \{1,2\}$ is the Hasse unit index of N and w_N is the number of complex roots of unity of N.

We need the following Proposition to be used throughout this paper.

Proposition 2.1.

1. *(See* [20, Lemma 2]*). Let K be a normal CM-field with Galois group G. Then the complex conjugation is in the centre of G.*
2. *(See* [20, Theorem 5]*). Let $k \subset K$ be two CM-fields. Assume that $[K:k]$ is odd. Then $Q_k = Q_K$ and h_k^- divides h_K^-.*
3. *(See* [18]*). Let K be a CM-field. If t prime ideals of K^+ are ramified in the quadratic extension K/K^+ then 2^{t-1} divides h_K^-.*
4. *(See* [20, Prop. 8]*). Let p be any odd prime number. Let $\mathbf{K}/\mathbf{M}$ be a cyclic extension of degree p of CM-fields and $\mathbf{K}^+/\mathbf{M}^+$ also be cyclic. Let t be the number of prime ideals of $\mathbf{M}^+$ which split $\mathbf{M}/\mathbf{M}^+$ and are ramified in $\mathbf{K}^+/\mathbf{M}^+$. Then $p^{t-1}h_{\mathbf{M}}^-$ divides $h_{\mathbf{K}}^-$, and $p^t h_{\mathbf{M}}^-$ divides $h_{\mathbf{K}}^-$ if p does not divide $w_{\mathbf{M}}$.*

3. Galois group type

3.1. **The abelian case.** In case where $\mathrm{Gal}(N/\mathbb{Q})$ is abelian, it was solved by Yamamura (see [33], 1994). To solve the class number one problem, first of all one has to deal with the construction of fields and the computation of their relative class number. In case of abelian fields, its construction comes from using groups of primitive Dirichlet characters [7] and the computation of their relative class numbers is also easy by using generalized Bernoulli numbers (see [32, Chapters 3 & 4]):

$$h_N^- = Q_N w_N \prod_{\chi\ odd}(-\frac{1}{2}B_{1,\chi}) \qquad \text{and} \qquad B_{1,\chi} = \frac{1}{f_\chi}\sum_{a=1}^{f_\chi}\chi(a)a$$

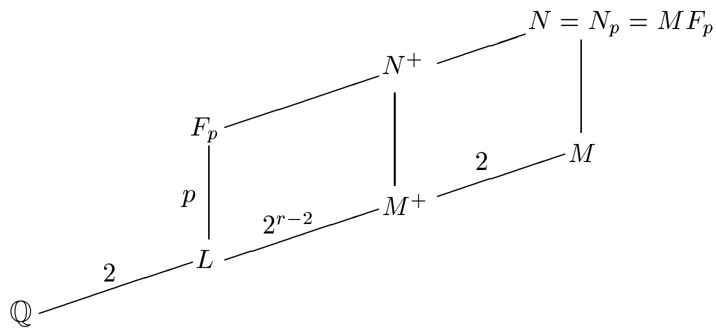
where χ is a primitive Dirichlet character of $N/\mathbb{Q}$ and f_χ is the conductor of χ.

Theorem 3.1. *(see* [33]*). There are exactly 172 imaginary abelian number fields with class number one. Their degrees are less than or equal to 24.*

Note that for imaginary abelian number fields the relative class number one problem has been lately solved by Chang and Kwon (see [2]).

3.2. **The dihedral case.** The first non-abelian group is naturally the dihedral group. Let N be a dihedral CM-field of degree $2n$ and $\mathrm{Gal}(N/\mathbb{Q}) = \langle \sigma, \tau : \sigma^n = 1, \tau^2 = 1, \tau^{-1}\sigma\tau = \sigma^{-1}\rangle$. Since the complex conjugation must be in the center of

$\mathrm{Gal}(N/\mathbb{Q})$, $n = 2m$ is even. For convenience, we deal with $n = 2^r p$ where p is any odd prime divisor and $r \geq 2$. We have the following lattice of subfields:



where

(i) L is a real quadratic field;
(ii) N, F_p and M are dihedral CM-fields, cyclic over L and of degrees $2n$, $2p$ and 2^r respectively (notice that if $r = 2$ then $M/\mathbb{Q}$ is biquadratic bicyclic);
(iii) N^+ and M^+ are the maximal totally real subfields of N and M, respectively.

It was S. Louboutin and R. Okazaki who have solved the class one problem for non-abelian normal octic CM-field (see [18], 1994). Moreover, they have settled the relative class number one problem for dihedral CM-fields of degree 2 power(see [19], 1998):

$$h_M^- = 1 \Rightarrow \begin{cases} M^+/L & \text{unramified} \\ [M:\mathbb{Q}] = & 4\ \ (147\ \text{ such fields}), 8\ \ (19\ \text{ such fields})\ \text{ or} \\ & 16\ \ (5\ \text{ such fields}). \end{cases}$$

According to Proposition 2.1, we have:

$$h_N = 1 \Rightarrow h_N^- = 1 \Rightarrow h_M^- = 1.$$

Therefore, the determination of dihedral CM-fields of degree 2 power with $h_M^- = 1$ gives us the list of M's as an candidate for $h_N = 1$.

1. Low bounds for h_N^- For finitely many candidates M, we verify that it holds $\zeta_M(s) \leq 0$ for $0 < s < 1$. If it does, using the following Theorem 3.2 and the formula (1) we could obtain good lower bounds for the relative class number of N (see [11], [14], [15] and [9]).

Theorem 3.2. 1. ([11, Prop. A]) *If $\zeta_N(\beta) \leq 0$ for $\beta \in [1 - 2/\log(d_n), 1[$, then*

$$Res_{s=1}(\zeta_N) \geq \frac{\epsilon_N}{e}(1-\beta),$$

where $\epsilon_{\mathbf{N}} = \max\{\frac{2}{5}\exp\left(-2\pi n/d_{\mathbf{K}}^{1/2n}\right), \epsilon''_{\mathbf{K}} = 1 - (2\pi n e^{1/n}/d_{\mathbf{K}}^{1/2n})\}$.

2. ([15, Theorem 11]) *Let N/k be an abelian extension of degree m that is unramified at all the infinite places. Then there exists a constant $\mu_k \geq 0$ such that*

$$Res_{s=1}(\zeta_N) \leq (Res_{s=1}(\zeta_k))^m \left(\frac{1}{2(m-1)}\log(d_N/d_k^m) + 2\mu_k\right)^{m-1}.$$

Moreover, if k is a real abelian field of degree $l \geq 2$ and conductor f_k, then

$$\mu_k Res_{s=1}(\zeta_k) \leq \frac{l-1}{2^{l+1}}(\log(f_k) + 2\mu_{\mathbb{Q}})^l,$$

where $\mu_{\mathbb{Q}} = (2 + \gamma - \log(4\pi))/2 = 0.023..$, γ is a Euler's constant.

For more details, we refer the reader to [19], [20] and [9].

2. Construction We have to construct the dihedral CM-field N. This construction of N is reduced to that of the real dihedral fields F_p of degree $2p$ which are cyclic over the fixed real quadratic field L. By using class field theory, this construction boils down to the construction of some groups of primitive characters on certain ray class groups of L (see [23]).

Let $\mathcal{M}$ be an integral ideal of a real quadratic field L. We let $I_L(\mathcal{M})$ denote the subgroup of the group I_L of fractional ideals of L generated by the integral ideals relatively prime to $\mathcal{M}$, and let $P_{L,Z}(\mathcal{M})$ be the group generated by the principal ideals of the form (α) where $\alpha \in A_L$ satisfies $\alpha \equiv a \pmod{\mathcal{M}}$ for some integer a relatively prime to $\mathcal{M}$. Then Class field theory provides us with so called the *ring class group* for the modulus $\mathcal{M}$, the quotient group $Cl_{L,Z}(\mathcal{M}) = I_L(\mathcal{M})/P_{L,Z}(\mathcal{M})$.

Theorem 3.3. *Let $\mathcal{M}$ be a given modulus of a real quadratic field L and assume that $\mathcal{M}$ is invariant under the action of* $\mathrm{Gal}(L/Q)$. *Then, there is a bijective correspondence between the real dihedral fields K of degree $2p$ containing L and such that the conductor $\mathcal{F}_{K/L}$ of the extension K/L is equal to $\mathcal{M}$ and the groups of order p generated by the primitive characters of order p on $Cl_{L,Z}(\mathcal{M})$.*

In particular, the conductor of the cyclic extension F_p/L is an integer f_p, i.e.

$$\mathcal{F}_{F_p/L} = f_p\mathcal{O}_L \text{ with } f_p \in \mathbb{Z}_{>0.}$$

Now we shall be dwelling upon a construction of all the primitive characters χ of order p on $Cl_{L,Z}(f)$. To begin with, write $h_L = p^{s_L(p)}h$ with $\gcd(p,h) = 1$, let $r_L(p)$ denote the p-rank of Cl_L, let

$$Cl_L^{(p)} = \prod_{i=1}^{r_L(p)} C_{p^{e_i}}$$

be the p-sylow subgroup of Cl_L and let J_i, $1 \leq i \leq r_L(p)$ be $r_L(p)$ integral ideals of A_L of norms relatively prime with f, whose ideal classes have order p^{e_i} such that $J_i^{p^{e_i}} = (\alpha_i)$ for $\alpha_i \in A_L$, respectively. Here, C_n denotes the cyclic group of order $n > 1$. Let now χ be a primitive character of order p on $Cl_{L,Z}(f)$. Then $\alpha \mapsto \chi((\alpha))$ defines a primitive character χ_0 of order p on $(A_L/(f))^*$, called the modular character associated to χ.

Proposition 3.4. *Fix some primitive character χ_0 of order p on $(A_L/(f))^*$ which is trivial on the fundamental unit ϵ_L of L and the image of Z.*

1. *If some $\chi_0(\alpha_i)$ is not equal to $+1$ for some $1 \leq i \leq r_L(p)$, then there is no primitive character χ on $Cl_{L,Z}(f)$ of order p whose associated modular character is equal to χ_0.*
2. *If all the $\chi_0(\alpha_i)$ are equal to $+1$ for $1 \leq i \leq r_L(p)$, then there are precisely $p^{r_L(p)}$ primitive characters χ on $Cl_{L,Z}(f)$ of order p whose associated modular characters are equal to χ_0.*

3. Computation of relative class number To compute h_N^-, we use the following formula instead of (1):

$$h_N^- = \frac{Q_N w_N}{(2\pi)^n}\sqrt{\frac{d_N}{d_{N^+}}}\prod_{\chi\in X_{N/L}^-} L(1,\chi)$$

where $X_{N/L}$ denotes the (Hecke) characters associated with the abelian extension N/L and $X_{N/L}^- = X_{N/L}\setminus X_{N^+/L}$. For a given primitive character χ on a ray class group of L, S. Louboutin has developed an efficient technique for computing values at $s=1$ of Hecke L-functions associated with χ which are useful to compute h_N^- (see [17]): For two positive real functions K_1 and K_2 such that $0\le K_2(B)\le K_1(B)\le 2e^{-B}$ for $B>0$, we have the following absolutely convergent series expansion

$$L(1,\chi) = \sum_{n\ge1}\frac{a_n(\chi)}{n}K_1(n/A_\chi) \quad + W_\chi\sum_{n\ge1}\frac{\overline{a_n(\chi)}}{n}K_2(n/A_\chi)$$

where $A_\chi = \sqrt{d_{\mathbf{L}} f_\chi/\pi^2}$, $a_n(\chi)=\sum_{N_{L/\mathbb{Q}}(\mathcal{I})=n}\chi(\mathcal{I})$ and W_χ denotes the Artin root number associated with this L-function. To solve the class number one problem, it remains to compute the class number h_{N^+} of the maximal real subfield. Finally, to compute h_{N^+} we use PARI- GP [1] or KANT [5] to obtain the following Theorem.

Theorem 3.5. *(See Y. Lefeuvre [9]). There are exactly 43 non-abelian dihedral CM-fields with relative class number one and 32 of them have class number one. Moreover all these fields have degree less than or equal to 24.*

4. The dicyclic case

For the dicyclic case, we don't need any computation to solve its class number one problem (see [20] and [16]). Let N be a dicyclic CM-field of degree $2n$. Since the complex conjugation must be in the centre of $\mathrm{Gal}(N/\mathbb{Q})$ (see Proposition 2.1), $n=2m$ is even. Let M be the maximal 2-subfield of N then M is a CM-field and h_M^- divides h_N^- (because $[N:M]$ is odd). Notice also that if $[M:\mathbb{Q}]=4$ then M is an imaginary cyclic quartic field, whereas if $[M:\mathbb{Q}]=2^r\ge 8$ then M is a dicyclic (quaternion) CM-field.

Theorem 4.1.
1. *(See [20] and [26]). The relative class numbers h_N^- of dicyclic CM-fields N of degree $2^r\ge 8$ are even. Moreover, if $2^r\ge16$ then $h_N^-\equiv 0 \pmod 8$.*
2. *(See [21]). Let N be a dicyclic CM-field of degree $4p$. Then $h_N^-\ge 4$ and there are four dicyclic CM-fields of degree $4p$ such that $h_N^-=4$.*

According to Proposition 2.1 and Theorem 4.1.1, we get

$$h_N=1\Rightarrow h_N^-=1\Rightarrow h_M^-=1\Rightarrow [M:\mathbb{Q}]=4.$$

Thus $[N:\mathbb{Q}]=4m$ with m odd and N has a subfield N_p which is a dicyclic CM-field of degree $4p$ for some odd prime p dividing m. Therefore, Proposition 2.1 implies that

$$h_N=1\Rightarrow h_N^-=1\Rightarrow h_{N_p}^-=1$$

but it is impossible according to Theorem 4.1.2.

Theorem 4.2. *(See [16, Theorem 7]). There is no non-abelian dicyclic CM-field with (relative) class number one.*

5. Normal CM-fields of small degrees

Let N be a normal CM-field of degree $2n \leq 48$ with Galois group G. Then, N is abelian, dihedral, dicyclic or of degree 16, 24, 32, 36, 40, 42 or 48. Since we have just briefly dealt with abelian, dihedral and dicyclic cases, we will investigate the remaining cases. But note that in case of degree 16 or 32, it requires more different techniques not such as those of the others to solve the class number one problem. In fact, these cases are rather difficult and complicated to deal with because they are 2-powers. So no attempt has been made here to explain these cases. For the moment, only the degree 16 is solved (see [12] and [19]). In degree 32, the problem is only solved for the normal CM-fields which are composita of two normal CM-fields of degree 16 with the same maximal real subfield (see [34]).

Theorem 5.1.
1. *(See* [12] *and* [19]*). There are* 8 *non-abelian non-dihedral normal CM-field* N *of degree* 16 *with class number one.*
2. *(See* [34]*). There are exactly four normal CM-fields of degree* 32 *with relative class number one which are composita of two normal CM-fields of degree* 16 *with the same maximal real subfield .*

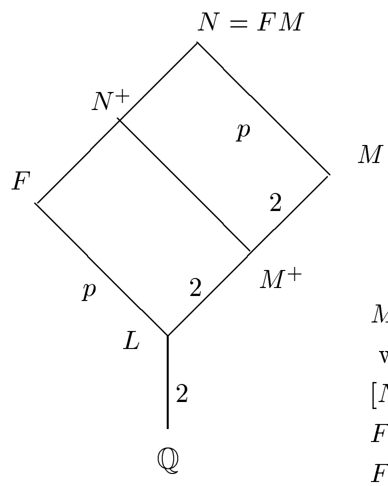
Now, let us consider the cases of degrees 24, 36, 40, 42 or 48. There are roughly two different techniques to be used for the determination of those fields with class number one (see [10], [3], [25], [22] and [4]). But, in degree 24 there are two groups $\mathcal{A}_4 \times C_2$ and $\mathrm{SL}_2(\mathbb{F}_3)$ for which these two techniques are useless. For these two groups we refer the reader to [10] because it's too hard to explain shortly.

5.1. $N = MF$ **where** M **is a normal CM-field with** $[N : M]$ **odd.** Let $N = MF$ be a compositum of M and F where M is a normal CM-field with $[N : M]$ odd. Then according to Proposition 2.1, we get

$$h_N = 1 \Rightarrow h_N^- = 1 \Rightarrow h_M^- = 1.$$

Since the degree of M is less than that of N, we can assume that the relative class number one problem for M has been already determined. Thus, M is one of the finitely many candidates which have relative class number one; in fact, the number of these candidates is not large. We fix one such M. Now we need the construction of the subfield F to get that of $N = FM$. If F is an abelian extension of a real subfield L of M^+, we can use the class field theory to construct F like that of dihedral fields. This technique is used in degrees 24 (see [16, corollary 6] and [25, Theorem 13]), 40 (see [25, Theorems 15 and 20]), 42 (see [22]) and 48(see[4]). In particular, if F is also a normal CM-subfield and $[N : F]$ is odd, we just have to look up in the list of the normal CM-fields F and M with relative class number one, which easily yields the determination of $h_N^- = 1$. This technique is used in degrees 24 (see [25, §1.1]), 36(see [3]) and 40 (see [25, §1.2]).

For example, let $\mathbf{N}$ be a normal CM-field of degree $8p$ with Galois group $G(\mathbf{N}/\mathbb{Q}) \simeq C_4 \times D_{2p}$. Then we have the following lattice of subfields :



M is an imaginary abelian field
with $G(M/\mathbb{Q}) = C_4 \times C_2$
$[N : M]$ is odd and M/L is cyclic
F is a dihedral real field
F/L is cyclic of degree p

$$h_N = 1 \Rightarrow h_N^- = 1 \Rightarrow h_M^- = 1.$$

There are six imaginary abelian octic fields M with Galois group $G(M/\mathbb{Q}) = C_4 \times C_2$ which are cyclic over a real quadratic subfield L and $h_M^- = 1$ (see [2]). For a fixed M, we use Theorem 3.2 to get an upper bound of the conductors of the cyclic extension F/L. Next, we construct F as dihedral case and we use Proposition 2.1 to alleviate the amount of relative class numbers computation required to settle. Finally, the computations of h_N^- and h_{N^+} show that there is only one such CM-field N with $h_N^- = 1$, and this field has class number one.

5.2. $N = N_1 N_2$ **is a compositum of two normal CM-fields** N_1 **and** N_2 **with the same maximal totally real subfield.** Let N be a compositum of two normal CM-fields N_1 and N_2 with the same maximal totally real subfield $N_1^+ = N_2^+ = K$ such that N/K is a biquadratic bicyclic extension. Then we have:

$$h_N^- = \frac{Q_N}{Q_{N_1} Q_{N_2}} \frac{w_N}{w_{N_1} w_{N_2}} h_{N_1}^- h_{N_2}^-$$

and $h_{N_1}^- h_{N_2}^-$ divides $4h_N^-$ (see [19, Proposition 2]). So $h_N^- = 1$ implies that $h_{N_1}^- \leq 4$ and $h_{N_2}^- \leq 4$. As the subsection 5.1, if we have settled the determination of N_1 and N_2 with $h_{N_1}^- \leq 4$ and $h_{N_2}^- \leq 4$, the list of these fields and the structure of $\mathrm{Gal}(N/\mathbb{Q})$ enable us to determine whether $h_N^- = 1$ or not. This technique is used in degrees 24 and 40 (see [25, Proposition 5 and Theorem 7]).

Finally, we have the following Theorem.

Theorem 5.2. 1. *(See* [10] *and* [25]*). There are* 6 *non-abelian non-dihedral normal CM-fields* N *of degree* 24 *with class number one.*

2. *(See* [3]*). There are* 3 *non-abelian normal CM-fields* N *of degree* 36 *with relative class number one and all of them have class number one.*
3. *(See* [25]*). There is only one non-abelian normal CM-field* N *of degree* 40 *with class number one.*

4. *(See [22]). There is no non-abelian normal CM-field N of degree 42 with relative class number one.*
5. *(See [4]). There is only one non-abelian normal CM-field N of degree 48 with class number one which has a normal CM-field of degree 16.*

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