Trends in Mathematics
Information Center for Mathematical Sciences
Volume 3, December 2000, Pages 82–86

FINITE ORDER VANISHING PROPERTY OF THE BERMGAN KERNEL FUNCTION

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ABSTRACT. We survey about the finite order vanishing property of the Bergman kernel function of a C^∞ bounded domain in $\mathbb{C}^n$. More precisely, given a C^∞ bounded domain Ω in $\mathbb{C}^n$ and $z_0 \in b\Omega, w_0 \in \overline{\Omega}$, we would like to know the possibility that the Bergman kernel $K(z,w)$ of Ω has the property

$$\frac{\partial^{|\alpha|+|\beta|}}{\partial z^\alpha \partial \overline{w}^\beta} K(z_0, w_0) \neq 0.$$

This property of the Bergman kernel is related to the boundary regularity of biholomorphic mappings between domains in $\mathbb{C}^n$.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{C}^n$. Let $H^2(\Omega)$ be the subspace of $L^2(\Omega)$ which consists of holomorphic functions in Ω. The space $H^2(\Omega)$, which is called the Bergman space of Ω, is a Hilbert space with the usual inner product

$$< f, g >= \int_\Omega f\overline{g}\, dV.$$

In fact, it is easy from the mean value property of harmonic functions to see that given a compact subset K of Ω there exists a constant C_K such that

$$\sup_K |f(z)| \leq C_K \|f\|_{L^2(\Omega)}, \quad \text{for all } f \in H^2(\Omega). \tag{1.1}$$

Here dV denotes the volume measure. Let $z \in \Omega$. It follows from (1.1) that the evaluation map φ_z of z defined by $\varphi_z(f) = f(z)$ is a bounded linear function from $H^2(\Omega)$ into $\mathbb{C}$. Then by the Riesz representation theorem there exists a function $K(\cdot, z) \in H^2(\Omega)$ such that for $f \in H^2(\Omega)$

$$f(z) = \varphi_z(f) =< f, K(\cdot, z) > .$$

The function $K(z,w)$ defined as above is called the Bergman kernel associated to the domain Ω. By definition, the Bergman kernel function $K(z,w)$ is in $H^2(\Omega)$ as a function of the variable z for fixed $w \in \Omega$ and satisfies the reproducing property:

2000 *Mathematics Subject Classification.* 32-02, 32H40, 32W05, 35N15, 32A25.
Key words and phrases. Bergman kernel, $\bar{\partial}$-Neuman problem, unique continuation.
This work was supported by Korea Research Foundation Grant(KRF-2000-015-DP0024). This study was also financially supported by Chonnam National University in the program, 2000, Math. Research Center(2000) of CNU, and BK21-project.
Received August 31, 2000.

for all $f \in H^2(\Omega)$,

$$f(z) = \int_\Omega \overline{K(w,z)} f(w)\ dV_w. \tag{1.2}$$

Since $K(\cdot, z) \in H^2(\Omega)$,

$$\begin{aligned} K(w,z) &= < K(\cdot,z), K(\cdot,w) > \\ &= \int_\Omega K(\zeta,z)\overline{K(\zeta,w)}\ dV_\zeta \\ &= \overline{\int_\Omega \overline{K(\zeta,z)}K(\zeta,w)\ dV_\zeta} \\ &= \overline{K(z,w)}. \end{aligned}$$

Hence we can rewrite the formula (1.2) as

$$f(z) = \int_\Omega K(z,w) f(w)\ dV_w.$$

On the other hand, if $\{\phi_j\}$ is a complete orthonormal basis for $H^2(\Omega)$ (such a basis exists because $H^2(\Omega)$ is separable), then it is easy from the Riesz-Fisher theory to see that the Bergman kernel function $K(z,w)$ is written in terms of ϕ_j's as follows:

$$K(z,w) = \sum_{j=1}^{\infty} \phi_j(z)\overline{\phi_j(w)}.$$

For instance, for the case of the unit disc $\Omega = \{z \in \mathbb{C} \mid |z| < 1\}$ in the plane, since the set $\{\phi_j(z) = \sqrt{\frac{j}{\pi}}\ z^{j-1}\}$ is a complete orthonormal basis, we obtain from an easy calculation that

$$K(z,w) = \frac{1}{\pi(1 - z\overline{w})^2}.$$

One of important roles in the direction of this survey which the Bergman kernel plays is that the Bergman kernel function is heavily related to the boundary regularity of biholomorphc mappings. Let's give an application relating the Bergman kernel to the boundary regularity. We say that a smooth bounded domain Ω in $\mathbb{C}^n$ satifies the *global regularity condition* if given a multi-index α there exist constants C_α and m_α such that

$$\sup_{z\in\Omega}\left|\frac{\partial^\alpha}{\partial z^\alpha}K(z,w)\right| \le C_\alpha \frac{1}{\operatorname{dist}(w, b\Omega)^{m_\alpha}} \quad \forall\ w \in \Omega.$$

(cf. [BB81]) It is known (see [Koh64]) that a smooth bounded strictly pseudoconvex domain satisfies the global regularity condition. Bell and Boas[BB81] proved the following: Suppose that Ω_1 and Ω_2 are smooth bounded pseudoconvex domains in $\mathbb{C}^n$ and let φ be a biholomorphic mapping between Ω_1 and Ω_2. If at least one of Ω_1 and Ω_2 satisfies the global regularity condition, then the map φ extends to a C^∞ diffeomorphism of $\overline{\Omega_1}$ to $\overline{\Omega_2}$. Hence a smooth bounded weakly pseudoconvex domain cannot be biholomorphically equivalent to a smooth bounded strictly pseudoconvex domain. This is a kind of classification result for domains in $\mathbb{C}^n$.

2. Finite Order Nonvanishing Property of the Bergman kernel

In this section, we consider the extent of order of what the Bergman kernel vanishes. Suppose that Ω is a C^∞ smooth bounded domain in $\mathbb{C}^n$. We suppose that for the simplicity, the Bergman kernel $K_\Omega(z,w)$ of Ω is in $C^\infty(\overline{\Omega}\times\overline{\Omega}\setminus\Delta_{b\Omega})$. (Many domains in $\mathbb{C}^n$ have this property, for instance, pseudoconvex domains of D'Angelo's finite type [D'A82].)

Now the question is "can the Bergman kernel $K(z,w)$ of Ω vanish at a point of (1): $\Omega\times\Omega$? Or of (2): $b\Omega\times\overline{\Omega}$?". In particular, we are interested in the case (2) because it is related to the boundary regularity of biholomorphic mappings as in the following. We say that a C^∞ bounded domain D satisfies the "condition B" if for any fixed $z_0\in b\Omega$, there exist $(n+1)$ points $w_0, w_1, \cdots, w_n \in \Omega$ such that

$$(i)\ K(z_0,w_0)\neq 0 \quad \text{and}$$

$$(ii)\ \det\begin{pmatrix} K(z_0,w_j) \\ \dfrac{\partial K}{\partial z_i}(z_0,w_j)\end{pmatrix}_{\substack{1\le i\le n\\ 0\le j\le n}} \neq 0.$$

Suppose that Ω_1 and Ω_2 are C^∞ bounded domains in $\mathbb{C}^n$ satisfying the condition B. Let $\varphi:\Omega_1\to\Omega_2$ be a biholomorphic mapping. It has been known ([BL80]) that if $K_{\Omega_i}(z,w)$, $j=1,2$, are in $C^\infty(\Omega_j\times\overline{\Omega_j})$, then φ extends smoothly to the boundary of Ω_1.

Here are historical backgrounds about answering the above questions. For the case (1), if $n=1$, i.e., in the complex plane, it has been shown ([SY76]) that the kernel K_Ω has no zeroes if and only if the domain Ω is simply connected. Hence if $n\ge 2$, since the Bergman kernel of a product domain $\Omega=\Omega_1\times\Omega_2$ is the multiplication of two lower dimensional Bergman kernels K_{Ω_1} and K_{Ω_2}, there are many domains in $\mathbb{C}^n$ whose Bergman kernel do have zeroes. In fact, the subclass of domains Ω whose Bergman kernels have no zeroes in Ω is nowhere dense in the class of C^∞ bounded pseudoconvex domains with respect to "a suitable" topology. (cf. [Boa96]).

For the case (2), by observing a result by Catlin[Cat80] that if Ω is a C^∞ bounded pseudoconvex domain and $z_0\in\Omega$, then there are many holomorphic functions f on Ω which are smooth up to the boundary of Ω and vanish to ∞-order at z_0, we can more generally ask whether given $z_0\in b\Omega$, $w_0\in\overline{\Omega}$ and a multi-index β, or not there exists a multi-index α such that

$$\frac{\partial^{|\alpha|+|\beta|}}{\partial z^\alpha \overline{w}^\beta}K(z_0,w_0)\neq 0?$$

Notice that holomorphic functions cannot have the infinite order vanishing property at any point of inside the domain except for the trivial case. In one complex variable, the Bergman kernel can be written in terms of the Green's function. Hence we can show that, using the boundary regularities of the Green's function, if Ω is a smoothly bounded n-connected domain in $\mathbb{C}$ and if $w_0\in\Omega$ is sufficiently near to $b\Omega$, $K(\cdot,w_0)$ has exactly $(n-1)$ zeroes in Ω. Furthermore, $K(z,w)\neq 0$ on $b\Omega\times b\Omega\setminus\Delta_{b\Omega}$. (cf. [SY76], [Bel92], [Chu93]).

In order to make an improvement of known results for research in the future, we state here in details the problem of the higher dimensional case (2). In the process, we will know how the finite order vanishing property of the Bermgna kernel is related to the $\overline{\partial}$-Neumann problem. Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. It is obvious from (1.1) that $H^2(\Omega)$ is a closed subspace of $L^2(\Omega)$ and hence there is the orthogonal projection $P : L^2(\Omega) \to H^2(\Omega)$ which is called the Bergman projection of Ω. In fact, we have for $h \in L^2(\Omega)$

$$Ph(z) = \int_{\Omega} K(z,w)h(w)dV_w.$$

Let $w_0 \in \Omega$ and let $d = \text{dist}(w_0, b\Omega)$. Choose $\chi \in C_0^\infty(B(0;1))$ which is radially symmetric about the origin with the normalized condition $\int \chi = 1$. Let χ_{w_0} be the compactly supported function near w_0 defined by $\chi_{w_0} = d^{-2n}\chi\left(\frac{\cdot - w_0}{d}\right)$. Then it follows from the polar coordinate representation that

$$K(\cdot, w_0) = P(\chi_{w_0}).$$

Furthermore,

$$\frac{\partial^{|\beta|}}{\partial \overline{w}^\beta} K(z,w_0) = P\left((-1)^{|\beta|}\frac{\partial^{|\beta|}}{\partial \overline{\zeta}^\beta}\chi_{w_0}(\zeta)\right)(z).$$

Hence in order to study about the finite order vanishing property of the Bermgan kernel, it is natural to consider the vanishing property of the Bergman projection P too. In the sequel, we want to find a condition for the domain Ω under which $P\phi$ cannot vanish to infinite order at $z_0 \in b\Omega$ where $\phi = (-1)^{|\beta|}\frac{\partial^{|\beta|}}{\partial \overline{\zeta}^\beta}\chi_{w_0}(\zeta)$.

Now let's assume that $P\phi \in C^\infty(\overline{\Omega})$. (This condition always holds if Ω satisfies the global regularity condition.) Let ϑ be the formal adjoint of $\overline{\partial}$ defined on $(0,1)$-forms. Then if $u \in C^\infty_{0,1}(\overline{\Omega})$ vanishes on the boundary of Ω, ϑu is orthogonal to the space $H^2(\Omega)$ by the Stoke theorem. On the other hand, Rosay proved the converse is true. (cf. [Ros82], [Bel93]). Hence, since $P\phi - \phi$ is orthogonal to $H^2(\Omega)$, there is $\alpha \in C^\infty_{0,1}(\overline{\Omega})$ such that the coefficients of α vanish on $b\Omega$ satisfying $\vartheta\alpha = P\phi - \phi$. Since $\vartheta\alpha = P\phi - \phi$ is orthogonal to $H^2(\Omega)$, it follows that $P(\vartheta\alpha) = 0$ in Ω. But the Kohn's formula $P = I - \vartheta N \overline{\partial}$ for the $\overline{\partial}$-Neumann operator N ([Koh63], [Koh64], [Koh84]) gives the identity $\vartheta\alpha = \vartheta N\overline{\partial}(\vartheta\alpha)$ on Ω. Thus we have an important identity

$$\vartheta\alpha = P\phi - \phi = \vartheta N\overline{\partial}(\vartheta\alpha) \quad \text{on } \Omega. \tag{2.1}$$

Remember that we want to ask if $P\phi$ cannot vanish to ∞-order at z_0. One answer for this question would be affirmative if $P\phi$ extends holomorphically to a full neighborhood of z_0. For example, this holds if the domain in question satisfies the "condition Q". (cf. [?]). To get an answer more generally, notice that from (2.1), since ϕ is compactly supported in Ω, $\vartheta\alpha$ is holomorphic in Ω near z_0. Hence $\overline{\partial}(\vartheta\alpha)$ vanishes near z_0. Thus the answer would be also "Yes" if the $\overline{\partial}$-Neumann operator N is globally real analytic hypoelliptic at z_0. (This condition for example holds if $b\Omega$ is real analytic near z_0 and z_0 is a strictly pseudoconvex point. (cf. [Tar78],[?]). For then $\vartheta N\overline{\partial}(\vartheta\alpha)$ would extend real analytically to a neighborhood of z_0 and then $P\phi$ extends holomorphically to a neighborhood of z_0.

Remark: This article is based on the paper[Bel93]. In fact, there is another result (which is a "local version") about the problem. Bell also pointed out in [Bel93] that one might "not" need the global analytic hypoellipticity of $\overline{\partial}$-Neumann problem to get an affirmative answer for our question. The author is currently doing research in this direction.

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