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JET PARAMETRIZATION OF CR MAPS BETWEEN REAL ANALYTIC CR MANIFOLDS

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Abstract. Let M and M' be germs of real analytic generic CR manifolds in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively. We show that if M is of finite type in the sense of Bloom-Graham and M' is essentially finite, then there exists a jet parametrization of CR mappings between M and M' under certain generic conditions.

Let M be a real analytic(C^ω) real manifold through the origin in $\mathbb{C}^N$. M is called a generic CR manifold of codimension d if there is a d-tuple of real valued C^ω functions $(\rho_1, \cdots, \rho_d)$ such that

$$M = \{Z \in \mathbb{C}^N : \rho_1(Z) = \cdots = \rho_d(Z) = 0\}$$

and $\partial\rho_1 \wedge \cdots \wedge \partial\rho_d \neq 0$ on M. The CR structure of M is defined by

$$T^{1,0}(M) := T^{1,0}(\mathbb{C}^N) \cap T(M),$$

where $T^{1,0}(\mathbb{C}^N)$ is the holomorphic tangent bundle over $\mathbb{C}^N$.

Now let M' be another C^ω generic CR manifold through $0 \in \mathbb{C}^{N'}$ and let f be a holomorphic map near the origin. We say that f is admissible if f maps M into M' and

$$df(T_0^{1,0}(M)) = T_{f(0)}^{1,0}(M').$$

If M and M' are real hypersurfaces of nondegenerate Levi form and f is invertible at 0, then it follows from Chern-Moser theory that f is determined by 2-jet at 0([CM]). Moreover, f depends analytically on its two jet at 0.

Later, finite determination and analytic dependence of CR mappings between C^ω CR manifolds on their finite jet at a reference point was studied by the method of Segre variety arguments([BER1],[BER2],[Z]) or by the method of prolongations of CR embedding equations([H1],[H2], [Ha]) under certain nondegeneracy conditions on the Levi form of M.

Recently in [BER3], Baouendi, Ebenfelt and Rothschild proved that a C^ω CR map f between M and M' is determined by a finite jet at 0 if M is of finite type in the sense of Bloom-Graham, essentially finite at 0 and f is of finite multiplicity.

In this paper we study the analytic dependence of admissible maps on their finite jet at 0 under certain generic conditions on M and M'.

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Let

$$M' = \{Z' \in \mathbb{C}^{N'} : \rho'_1(Z') = \cdots = \rho'_{d'} = (Z') = 0\},$$

where ρ'_l, $l = 1, ..., d'$, are C^ω real valued functions such that $\partial\rho'_1 \wedge \cdots \wedge \partial\rho'_{d'} \neq 0$ on M'. The complexification $\mathcal{M}'$ of M' is defined by

$$\mathcal{M}' = \{(Z', \xi') \in U' \times \overline{U}' : \rho_1(Z', \xi') = \cdots = \rho_{d'}(Z', \xi') = 0\},$$

where U' is a sufficiently small neighborhood of 0 in $\mathbb{C}^{N'}$. Now fix $\overline{p} \in \overline{U}'$. The Segre variety Q_p is an $(N' - d')$-dimensional complex manifold defined by

$$Q_p = \{Z' \in U' : (Z', \overline{p}) \in \mathcal{M}'\}.$$

Choose a C^ω basis $\{L'_1, \cdots, L'_{n'}\}$ of $T^{1,0}(M')$, where $n' = N' - d'$. We write

$$L'_j = \sum_{k=1}^{N'} a_{jk}(Z', \overline{Z}') \frac{\partial}{\partial Z_k}.$$

Set

$$X'_j = \sum_{k=1}^{N'} \overline{a}_{jk}(\xi', 0) \frac{\partial}{\partial \xi_k},\ \xi' \in \overline{U}',$$

where $\overline{a}_{jk}(\xi', 0) = \overline{a_{jk}(\overline{\xi}', 0)}$.

For an n'-tuple of nonnegative integers $\alpha = (a_1, \cdots, a_{n'})$, define

$$c_{\alpha,l}(Z') = {X'}^{\alpha} \rho_l(Z', \xi')\Big|_{\xi' = 0},$$

where ${X'}^{\alpha} = {X'}_1^{a_1} \cdots {X'}_{n'}^{a_{n'}}$.

Definition 1. M' is said to be essentially finite at 0 if the $\mathbb{C}$-vector space $\mathcal{O}[Z']/\left(c_{\alpha,l}(Z')\right)$ is of finite dimension, where $\mathcal{O}[Z']$ is the ring of germs of holomorphic functions at 0 and $(c_{\alpha,l}(Z'))$ is the ideal generated by $\{c_{\alpha,l}(Z') : \alpha \in \mathbb{Z}_+^{n'}\}$ in $\mathcal{O}[Z']$.

Denote by $j^K_{Z'} Q_p$ the K-jet of Q_p at Z' and by $G^K_{n'}(U')$ the bundle of K-jets of n'-dimensional complex manifolds in U'.

Suppose that M' is essentially finite at 0. Since $\mathcal{O}[Z']$ is a Noetherian ring, there exists a positive integer K_0 such that $(c_{\alpha,l}(Z')) = (c_{\alpha,l}(Z') : |\alpha| \leq K_0)$. Now define $\pi : \mathcal{M}' \to G^{K_0}_{n'}(U')$ by

$$\pi(Z', \xi') = (Z', j^{K_0}_{Z'} Q_{\overline{\xi}'}).$$

Lemma 2. *$\pi : \mathcal{M}' \to G^{K_0}_{n}(U')$ is a germ of a finite branched holomorphic covering onto its image.*

Let U and U' be open neighborhoods of 0 in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively. By $j^K_Z f$ denote the K-jet of holomorphic map f at Z. Also by $J^K_Z(U, U')$ denote the K-jet space of holomorphic maps f into U' at $Z \in U$ and by $J^K(U, U')$ the K-jet bundle of holomorphic maps.

Theorem 3. *Let M and M' be C^ω generic CR manifolds through 0 in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively and let f be a germ of an admissible map between M and M' such that $f(0) = 0$.*

Suppose M is of finite type at 0 in the sense of Bloom-Graham and M' is essentially finite at 0. Suppose further that there exists an integer K_0 such that the critical set of the map π is of codimension greater than or equal to 2 at 0.

Then there exist open neighborhoods U of 0 in $\mathbb{C}^N$, an open neighborhood V of $j_0^K f$ in $J_0^K(U,U')$ for a certain positive integer K and a holomorphic map Φ on $U \times V$ such that for every germ of a holomorphic map h at 0 with $j_0^K h \in V$ and $h(M) \subset M'$,

$$h(Z) = \Phi(Z, j_0^K h)$$

for all Z sufficiently close to 0.

The estimate of the integer K is not known in general.

We use the method of Segre varieties. First we establish a basic identity of reflection type on M under the condition that M' is essentially finite at 0 and codimension of the critical set of π is greater than or equal to 2. This identity is given by meromorphic functions Ψ^i, $i = 1, ..., N'$, defined on a variety of $\mathcal{M} \times_U J^K(U,U')$, where $\mathcal{M}$ is the complexification of M(see §1). Then we construct holomorphic functions Φ^i, $i = 1, ..., N'$, which coincide with Ψ^i if h maps M into M'. By iterations of this basic identity as in [BER2] and [Z], we prove the theorem.

In [BER3], f need not be an admissible map. Assume that M and M' have the same CR dimension, i.e. $N - d = N' - d'$. We say that f is of finite multiplication at 0 if $f : Q_0 \to Q_{f(0)}$ is a finite branched holomorphic covering at 0. If M is of finite type in the sense of Bloom-Graham, M' is essentially finite and f is of finite multiplicity at 0, then f is determined by a finite jet at a point. Then

Question 4. Let M and M' be as in Theorem 3. Suppose M and M' have the same CR dimension. Suppose further that f is of finite multiplicity at 0. Does there exist a jet parametrization of CR mappings between M and M'? If so, can one determine the integer K?

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