Trends in Mathematics
Information Center for Mathematical Sciences
Volume 3, December 2000, Pages 73–77

DISTORTION THEOREMS IN ONE AND SEVERAL VARIABLES

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ABSTRACT. Distortion theorems for certain classes of univalent functions of the unit disk to the complex plane $\mathbb{C}$ have been established and expressed into two-point distortion theorems(comparison theorems) for various regions in $\mathbb{C}$ in terms of the hyperbolic metric on the regions. In several variables some analogous results have been established. In this article, we survey these distortion theorems in one and several variables.

1. INTRODUCTION

Let S be the class of functions f analytic and univalent in the unit disk $\mathbb{D}$, normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Thus each $f \in S$ has a Taylor series expansion of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots, \quad |z| < 1$. In 1907, Koebe showed that the range of every function $f \in S$ contains the disk$\{|w| < c\}$ for an absolute constant c. The value $c = \frac{1}{4}$ was determined by Bieberbach by proving $|a_2| \leq 2$ a few years later. Bieberbach's inequality $|a_2| \leq 2$ has several implications in the geometry of conformal mapping. One important consequence is the Koebe distortion theorem [D]. The term "distortion" comes from the geometric interpretation of $|f'(z)|$ as the infinitesimal magnification factor of arclength under f. The distortion theorem can be applied to obtain the next sharp upper and lower bounds for $|f(z)|$ [D].

Theorem 1. *(Growth Theorem) For each $f \in S$,*

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}, \quad |z| < 1.$$

For each $z \in \mathbb{D}, z \neq 0$, equality occurs if and only if f is a suitable rotation of the Koebe function, $K(z) = z/(1-z)^2$.

The study of "coefficient estimates" for certain classes of univalent functions on $\mathbb{D}$ has brought distortion and growth theorems for the classes.

In this article, first we show that the hyperbolic metric was employed to establish distortion theorems for convex functions, spherically convex functions, and hyperbolically convex functions. We also give characterizations for the related domains in $\mathbb{C}$ with those functions. Then, we discuss about distortion theorems in several variables and introduce some results.

2000 *Mathematics Subject Classification.* Primary 30C45,30C25, Secondary 32F32.

Key words and phrases. distortion theorem, univalent functions, hyperbolic metric, hyperbolically convex function, spherical density, spherically convex region.

Received August 31, 2000.

2. Preliminaries

Hyperbolic metric: We recall basic facts about hyperbolic geometry. The hyperbolic metric on $\mathbb{D}$ is $\lambda_{\mathbb{D}}(z)|dz| = |dz|/(1-|z|^2)$. It has constant Gaussian curvature -4. A region Ω on the Riemann sphere $\mathbb{P}$ is called hyperbolic if $\mathbb{P}\backslash\Omega$ contains at least three points. The density of the hyperbolic metric on a hyperbolic region $\Omega \subset \mathbb{P}$ is obtained from $\lambda_\Omega(f(z))|f'(z)| = 1/(1-|z|^2)$, where $f : \mathbb{D} \to \Omega$ is any meromorphic universal covering projection of $\mathbb{D}$ onto Ω. The distance function d_Ω on Ω is induced by the hyperbolic metric. The hyperbolic metric and the associated distance function are both conformal invariants.

Spherical metric: The spherical metric on $\mathbb{P}$ is the conformal metric $\lambda_{\mathbb{P}}(w)|dw| = |dw|/(1+|w|^2)$. It has constant Gaussian curvature $+4$ and it is invariant under rotations of the sphere. The distance function $d_{\mathbb{P}}$ is induced on $\mathbb{P}$ by the spherical metric. For a hyperbolic region Ω on $\mathbb{P}$, it is natural to consider the spherical density of the hyperbolic metric on Ω [M],

$$\mu_\Omega(w) = \frac{\lambda_\Omega(w)|dw|}{\lambda_{\mathbb{P}}(w)|dw|} = (1+|w|^2)\lambda_\Omega(w),$$

while we consider the hyperbolic density as the ratio of the hyperbolic metric on Ω by the euclidean metric on Ω which is inherited by the euclidean metric on $\mathbb{C}$.

Invariant distortion theorems: Kim and Minda [KM_1] obtained an invariant distortion theorem for univalent functions through Koebe growth theorem. The invariant distortion theorem characterizes univalence while Koebe growth theorem is a necessary condition for univalence.

Theorem 2. *(Invariant Koebe distortion theorem) Suppose f is univalent on $\mathbb{D}$. Then for all $a, b \in \mathbb{D}$,*

$$|f(a) - f(b)| \geq \frac{\sinh(2d_{\mathbb{D}}(a,b))}{2\exp(2d_{\mathbb{D}}(a,b))}\, max\{|D_1 f(a)|, |D_1 f(b)|\},$$

where $|D_1 f(z)| = |f'(z)|(1-|z|^2)$. Equality holds for distinct points $a, b \in \mathbb{D}$ if and only if $f = S \circ K \circ T$, where S and T are conformal automorphisms of $\mathbb{C}$ and $\mathbb{D}$, respectively, and a, b lie on the axis of symmetry of f. Conversely, if a nonconstant holomorphic function f satisfies the inequality, then f is univalent on $\mathbb{D}$.

They [KM_1] state Blatter's distortion theorem [B] in a more general form. Their statement gives the improved version of Blatter's theorem and a connection between Blatter's and this invariant theorem.

3. Distortion theorems in one variable

We call $f \in S$ a convex function if $f(\mathbb{D})$ is convex in $\mathbb{C}$. We know that the analogous distortion and growth theorems for normalized convex univalent functions on $\mathbb{D}$ can be established as for normalized univalent functions on $\mathbb{D}$ [D]. The distortion theorem has been developed into two-point distortion theorem for convex univalent function. This two-point distortion theorem can be expressed as a comparison theorem between the euclidean distance and the hyperbolic distance [KM_1]. Each of these theorems characterizes either convex univalent functions on $\mathbb{D}$ or convex hyperbolic regions in $\mathbb{C}$.

Theorem 3. *Suppose $\Omega \subset \mathbb{C}$ is a convex hyperbolic region. Then for all $A, B \in \Omega$,*

$$|A - B| \geq \frac{1}{2}\tanh(d_\Omega(A,B))\left((1/\lambda_\Omega(A)) + (1/\lambda_\Omega(B))\right).$$

Equality holds if and only if Ω is a half plane and A and B lie on a line perpendicular to the edge of the half-plane. Conversely, if Ω is a hyperbolic region in $\mathbb{C}$ and the preceding inequality holds for all $A, B \in \Omega$, then Ω is convex.

Corollary 1. *Suppose f is convex univalent on $\mathbb{D}$. Then for all $a, b \in \mathbb{D}$,*

$$|f(a) - f(b)| \geq \frac{1}{2}\tanh(d_{\mathbb{D}}(a,b))(|D_1 f(a)| + |D_1 f(b)|).$$

Equality holds for distinct $a, b \in \mathbb{D}$ if and only if $f = S \circ H \circ T$ where S and T are conformal automorphisms of $\mathbb{C}$ and $\mathbb{D}$, respectively, and $H(z) = z/(1-z)$, and a and b lie on any axis of symmetry of f. Conversely, if a nonconstant holomorphic function f defined on $\mathbb{D}$ satisfies this inequality, then f is convex univalent on $\mathbb{D}$.

Yamashita [Y] proved that a region $\Omega \subset \mathbb{C}$ is convex if and only if

$$|(1/\lambda_\Omega(A)) - (1/\lambda_\Omega(B))| \leq 2|A - B|$$

for all $A, B \in \Omega$. The next result is a spherical analog due to Kim and Minda [KM$_2$]. We first state definitions of spherical convexity.

Definitions A region $\Omega \subset \mathbb{P}$ is called spherically convex if for each pair of points $A, B \in \Omega$, any spherical geodesic connecting A and B also lies in Ω. See [M] for more information about spherical convexity. A meromorphic univalent function f on $\mathbb{D}$ is called spherically convex if the image $f(\mathbb{D})$ is spherically convex in $\mathbb{P}$.

Theorem 4. *Let $\Omega \subset \mathbb{P}$ be a hyperbolic region. Then Ω is spherically convex if and only if for all $A, B \in \Omega$,*

$$|\arcsin(1/\mu_\Omega(A)) - \arcsin(1/\mu_\Omega(B))| \leq 2d_{\mathbb{P}}(A,B).$$

Also, several characterizations for spherically convex regions $\Omega \subset \mathbb{P}$ are given in terms of the spherical density of the hyperbolic metric on Ω in [KM$_2$]. Other two-point distortion theorems for spherically convex functions and for spherically convex regions in terms of the hyperbolic metric are established in [MM$_1$]. The theorems in [MM$_1$] for spherically convex region gives the lower bound of the spherical distance between any two point in the region. So far, sharp upper bounds have not been established in either the two-point distortion theorem for spherically convex functions or the two-point comparison theorem for spherically convex regions. The sharp lower bound on $|f'(z)|$ is known for all $z \in \mathbb{D}$ while the sharp upper bound on $|f'(z)|$ has been established only for z near the origin [MMM].

Recently, hyperbolically convex function f on $\mathbb{D}$ has been studied by Ma, Minda [MM$_2$] and Mejia, Pommerenke [MP]. Several distortion theorems for these functions are given by them. The definitions of hyperbolically convexity is as follows.

Definitions A region $\Omega \subset \mathbb{D}$ is said to be hyperbolically convex if for each pair of points $a, b \in \Omega$, any hyperbolic geodesic(relative to the hyperbolic metric on $\mathbb{D}$) joining a and b also lies in Ω . We call $f : \mathbb{D} \to \mathbb{D}$ hyperbolically convex function, if $f(\mathbb{D})$ is a hyperbolically convex region in $\mathbb{D}$.

4. Distortion theorems in several variables

About seventy years ago, Henri Cartan suggested that geometric function theory of one complex variable should be extended to biholomorphic mappings of several complex variables. In particular, he cited the special classes of starlike functions and of convex functions as appropriate topics for generalization. He noted some difficulties of generalization and pointed out that growth theorem for univalent functions would not extend to the polydisc (nor to the ball). Also, he observed that there is no Koebe covering theorem for normalized biholomorphic mappings of the polydisc (or the ball).

Distortion theorems in several variables are closely related with the following two subjects; Let f be a normalized holomorphic mapping of the unit ball B^n into $\mathbb{C}^n$. f is said to be normalized if $f(0) = 0$ and $df(0) = I$, the identity matrix.

(1) To estimate the largest radius r of the ball of center 0 where f is univalent.

(2) To estimate the largest radius r of the ball of center 0 which is contained in $f(B^n)$.

In [BFG], versions of both the growth theorem and Koebe covering theorem for the class of normalized univalent starlike mappings on the unit ball were found. The following theorem is one of them.

Theorem 5. *Let f be a normalized univalent starlike mapping of B^n into $\mathbb{C}^n$. For any point $z \in B^n$,*

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

Furthermore, the statements are sharp.

Graham and Varolin determined the Bloch constant (equivalently the Koebe constant) for convex maps of B^n with $k-$fold symmetry, $k \geq 2$ [GV]. They also estimate and in some cases compute the Bloch constant for starlike maps of B^n with $k-$fold symmetry in the same paper. Graham [G] also employed the Roper-Suffridge extension operator which provides a way of extending a (locally) univalent function f (of $\mathbb{D}$ to $\mathbb{C}$) to a (locally) univalent map F of B^n to $\mathbb{C}^n$, and then showed that if f belongs to a class of univalent functions which satisfies a growth theorem and a distortion theorem, then F satisfies a growth theorem and consequently a covering theorem. He also obtained covering theorems : If f is convex, then the image of F contains a ball of radius $\pi/4$. If f is normalized univalent on $\mathbb{D}$, the image of F contains a ball of radius $1/2$.

Besides Graham and Varolin, Duren, Rudin, FitzGerald, Barnard, Gong, Minda, Pfaltzgraff have worked on distortion problem in several variables.

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