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GROWTH OF MEAN LIPSCHITZ FUNCTIONS IN THE COMPLEX BALL

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ABSTRACT. Holomorphic mean Lipschitz space in the open unit ball of $\mathbb{C}^n$ is introduced. The membership of the space is expressed in terms of the growth of the radial derivatives, which reduces to a well-known theorem when $n = 1$. The membership is also expressed in terms of the mean growth of the tangential derivatives.

I. INTRODUCTION

Let B be the open unit ball of $\mathbb{C}^n$ and Δ denote B when $n = 1$. Let S be the boundary of B. The surface area measure on S normalized to have total mass one will be denoted by σ.

For $0 < \alpha < 1$, we say that $f \in Lip_\alpha(B)$ if f is holomorphic in B, continuous in $\bar{B}$, and satisfies the Lipschitz condition of order α : $|f(z) - f(w)| = O(|z - w|^\alpha)$, $z, w \in \bar{B}$.

Let $H^p(B)$, $1 \leq p < \infty$, denote the Hardy space in B. It consists of those f holomorphic in B for which $\|f\|_{H^p} = \lim_{r\to 1} M_p(r, f) < \infty$, where

$$M_p(r,f) \;=\; \left(\int_S |f(r\zeta)|^p d\sigma(\zeta)\right)^{\frac{1}{p}}$$

if $0 < p < \infty$, and

$$M_\infty(r,f) = \sup_{\zeta\in S} |f(r\zeta)|.$$

When $n = 1$, there is the holomorphic mean Lipschitz space $Lip^p_\alpha(\Delta)$, $1 \leq p < \infty$, $0 < \alpha < 1$, which is a subspace of $H^p(\Delta)$ and is larger than $Lip_\alpha(\Delta)$. It is defined to consist of $f \in H^p(\Delta)$ satisfying the mean Lipschitz condition :

$$\left(\int_0^{2\pi} |f(e^{i\theta}) - f(e^{i(\theta+h)})|^p \,\frac{d\theta}{2\pi}\right)^{1/p} = O(|h|^\alpha). \tag{1}$$

See for example [D, Chapter 5].

In [K], a new definition of the holomorphic mean Lipschitz space in B was given, which extends naturally that of the one dimensional space. Then a characterization

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of the membership of the space in terms of the mean growth of the concerning derivatives was given, which generalizes a classical result of Hardy and Littlewood. The purpose of this talk is to introduce some related results on the growth of the mean Lipschitz functions in B. See [K] and [KK] for the precise results and their proofs.

II. Defining Mean Lipschitz Space

The definition of $Lip_\alpha^p(B)$ was adapted as follows.

Definition [K]. *For $0 < \alpha < 1$ and $1 \le p < \infty$, we say that $f \in Lip_\alpha^p(B)$ if $f \in H^p(B)$ and*

$$\left(\int_S |f(U\zeta) - f(\zeta)|^p \, d\sigma(\zeta)\right)^{\frac{1}{p}} = O(|h|^\alpha) \tag{2}$$

for all unitary operators U of $\mathbb{C}^n$, where $h = (h_1, h_2, ..., h_n) \in \mathbb{R}^n$ is determined to be that $e^{ih_1}, e^{ih_2}, ..., e^{ih_n}$ are the eigenvalues of U.

The right side quantity of (2) is $\left(\sum |h_j|^2\right)^{\alpha/2}$. When $n = 1$, this definition reduces to (1). $Lip_\alpha^p(B)$ is a Banach space with the norm

$$|f(0)| + sup \frac{1}{|h|^\alpha}\left(\int_S |f(U\zeta) - f(\zeta)|^p \, d\sigma(\zeta)\right)^{\frac{1}{p}},$$

where the supremum is taken with respect to the unitary operators U and h is determined as in (2).

III. Radial Derivative of the Lipschitz functions

As far as the Lipschitz functions are concerned, the most interesting and dominant result may be their relationship with the growth of derivatives. As is well-known, two theorems of Hardy and Littlewood express the membership of $Lip_\alpha(\Delta)$ and $Lip_\beta^p(\Delta)$ in terms of the growth of the derivatives:

Theorem 1 [Hardy-Littlewood]. *Let $0 < \alpha < 1$. If f is holomorphic in Δ, then*

$$f \in Lip_\alpha(\Delta) \iff M_\infty(r, f') = O\left((1-r)^{\alpha-1}\right). \tag{3}$$

Theorem 2 [Hardy-Littlewood]. *Let $0 < \alpha < 1$ and $1 \le p < \infty$. If f is holomorphic in Δ, then*

$$f \in Lip_\alpha^p(\Delta) \iff M_p(r, f') = O\left((1-r)^{\alpha-1}\right). \tag{4}$$

See [D, Theorem 5.1] and [D, Theorem 5.4].

In case $n > 1$, (3) holds with Rf in place of f', where Rf, the radial derivative of f, is defined by

$$Rf(r\zeta) \;=\; \left(\sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}\right)(r\zeta), \quad 0 < r < 1,\ \zeta \in S.$$

Theorem 3[R]. *Let $0 < \alpha < 1$. If f is holomorphic in B, then*

$$f \in Lip_\alpha(B) \quad \Longleftrightarrow \quad M_\infty(r, Rf) = O\left((1-r)^{\alpha-1}\right).$$

See [R, Theorem 6.4.9 and Theorem 6.4.10].

Using Rf in place of f', it is naturally called for to extend (4) to the case of $n \geq 1$. We have the following

Theorem 4[K]. *Let $0 < \alpha < 1$ and $1 \leq p < \infty$. If f is holomorphic in B, then*

$$f \in Lip_\alpha^p(B) \quad \Longleftrightarrow \quad M_p(r, Rf) = O\left((1-r)^{\alpha-1}\right).$$

IV. More on the growth of Mean Lipschitz functions

Define the tangential derivatives by

$$T_{ij} = \bar{z}_i \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial z_i}, \quad i, j = 1, 2, ...,$$

and let $T^k = \Pi_{l=1}^k T_{i_l j_l}$. As is well-known, the growth along tangential directions (when $n \geq 2$) are two times better :

Theorem 5 [KK]. *Let $0 < \alpha < 1/2$ and $1 \leq p < \infty$. If $n \geq 2$ and f is holomorphic in B, then the followings* (5) *and* (6) *are equivalent:*

$$f \in Lip_\alpha^p(B) \tag{5}$$

$$M_p(r, Tf) = O((1-r)^{\alpha - 1/2}) \tag{6}$$

See [KK] for the detailed proofs and more equivalent statements involved with T^k, the unitary operators and the non-isotrophic distance.

We believe that similar results hold for M-harmonic Lipschitz functions. See [AB] and [JP].

Our results here will be used in characterizing the composition operators from Bloch type spaces.

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