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A SURVEY ON MAJORIZATIONS AND ITS PRESERVERS

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ABSTRACT. The notion of majorization arises in a variety of contexts. In this article, we shall give a brief survey of recent work on majorization polytope and matrix version of majorization preservers.

1. INTRODUCTION

Majorization is a topic of much interest in various areas of mathematics and statistics. The definition of the majorization $\mathbf{x} \prec \mathbf{y}$ for real row vectors $\mathbf{x}$ and $\mathbf{y}$ with n components is motivated as a way of making precise the idea that the components of $\mathbf{x}$ are *less spread out than* the components of $\mathbf{x}$. If $\mathbf{x}$ and $\mathbf{y}$ are nonincreasing n-vectors of nonnegative real numbers such that $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$ for $k = 1, \ldots, n$ with equality at $k = n$, then we say that $\mathbf{x}$ is (vector) majorized by $\mathbf{y}$ and write $\mathbf{x} \prec \mathbf{y}$. In particular, majorization plays a significant role in matrix theory. For instance, majorization relations among eigenvalues and singular values of matrices produce a lots of norm inequalities. In this paper, some of recent results on this topic are summarized, and open problems will be posed.

In 1979, Marshall and Olkin listed the connection of majorization in analytic inequalities, combinatorics, matrix theory, numerical analysis, probability and statistics. Ando[3] also gave a revised survey on the research of the theory of majorization in 1994. It covers recent research on majorization related to eigenvalues and singular values, matrix inequalities, and norm inequalities. Recently Y. L. Tong of Johns Hopkins University introduced the possible applications of the theory of majorization to AIDS research on the spread of HIV virus (http://www.cs.jhu.edu/~cowen/seminars/tong.html) because if the heterogeneities of two possible contact vectors of a susceptible can be partially ordered via majorization, then the corresponding probability functions of escaping infection can be similarly ordered. In this article, we will give more emphasis on the results on majorization polytope and matrix version of majorization preservers.

Throughout this paper, let $M_{mn}(\mathbb{R})$ be the set of all $m \times n$ real matrices, $M_{mn}(\mathbb{R}^+)$ be the set of all $m \times n$ nonnegative real matrices, $M_n(\mathbb{R})$ be the set of all $n \times n$ real matrices, $\mathbf{P}(n)$ be the set of all $n \times n$ permutation matrices and $\mathbb{R}^n$ be the set of all real n-dimensional row vectors.

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Definition 1.1. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x}$ *is majorized by* $\mathbf{y}$, written $\mathbf{x} \prec \mathbf{y}$, if

$$\max\{\mathbf{x}\mathbf{v}^T | \mathbf{v} \in V_{k,n}\} \leq \max\{\mathbf{y}\mathbf{v}^T | \mathbf{v} \in V_{k,n}\}$$

for all $k = 1, \ldots, n$ and the equality holds when $k = n$, where $V_{k,n}$ denotes the set of all $1 \times n$ $(0,1)$-matrices whose entries have sum k.

It can be improved if we reorder the components in $\mathbf{x}$ and $\mathbf{y}$. In other words, for any $\mathbf{x} = [x_1, \ldots, x_n] \in \mathbb{R}^n$, let $x_{[1]} \geq \cdots \geq x_{[n]}$ be the components of $\mathbf{x}$ in the *decreasing order*, and let $\mathbf{x}_\downarrow = [x_{[1]}, \ldots, x_{[n]}]$ denote the decreasing rearrangement of $\mathbf{x}$.

Then the definition 1.1 can be replaced by the next definition.

Definition 1.2. For $\mathbf{x} = [x_1, \ldots, x_n]$, $\mathbf{y} = [y_1, \ldots, y_n] \in \mathbb{R}^n$, $\mathbf{x} \prec \mathbf{y}$ if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$$

for all $k = 1, \ldots, n$ and the equality holds when $k = n$.

Next two definitions are well-known and play an important role.

Definition 1.3. A nonnegative real matrix is *row stochastic* if all its row sums is equal to 1. And $\mathbf{RS}(n)$ denotes a set of all $n \times n$ row stochastic matrices.

Definition 1.4. A nonnegative real matrix is *doubly stochastic* if all its row sums and column sums are equal to 1. And $\mathbf{DS}(n)$ denotes a set of all $n \times n$ doubly stochastic matrices.

In 1946, Birkhoff gave an important characterization of the set of doubly stochastic matrices.

Theorem 1.5. [10] *The permutation matrices constitute the extreme points of the set of doubly stochastic matrices. Moreover, the set of all doubly stochastic matrices forms a convex polytope of the permutation matrices.*

In fact, every $n \times n$ doubly stochastic matrices can be represented as a convex combination of at most $n^2 - 2n + 2$ permutation matrices.

Our main interest is in the matrix version of majorization. The matrix version of majorization is that an $m \times n$ real matrix A is majorized by B if there is a certain matrix X such that $A = BX$. We are interested in a specific type of majorization of matrices which was given in [27]. One such possibility is to say that A is *multivariate majorized* by B if there exists a *doubly stochastic matrix* D such that $A = BD$. This is motivated by the theorem of Hardy-Littlewood and Polya saying that for row vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$, $\mathbf{a}$ is to be *majorized* by $\mathbf{b}$, if there exists an $n \times n$ doubly stochastic matrix D such that $\mathbf{a} = \mathbf{b}D$. Another possibility is to say that A is *matrix majorized* by B if there exists a *row stochastic matrix* R such that $A = BR$, which was defined in [16] by Dahl. The next definition was given in [9]. A is *directionally majorized* by B (written $A \prec_d B$) if $A\mathbf{v} \prec B\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$.

Now we give the formal definitions of matrix version of majorization.

Definition 1.6. [27] Let A and B be $m \times n$ matrices. Then A is said to be *multivariate majorized* by B, written $A \prec^{mul} B$, if $A = BD$ where $D \in \mathbf{DS}(n)$.

Definition 1.7. [16] Let A and B be $m \times n$ matrices. Then A is said to be *matrix majorized* by B, written $A \prec B$, if $A = BR$ where $R \in \mathbf{RS}(n)$.

We will introduce the characterization of majorization polytope in section 2, and the characterization of linear preservers on matrix versions of majorization in section 3.

2. Characterizations of majorization polytopes

Ando gave a nice survey on the majorization in [3], 1994. This include most of results on inequalities which was induced from the known fact of majorization. So now we list the recent results on the characterization of majorization polytopes in here.

Let $\mathbf{y}$ be majorized by $\mathbf{x}$. Several investigation of the polytope of doubly stochastic matrices D for which $\mathbf{y} = \mathbf{x}D$ has been given. And it were determined when there is a positive matrix D and when there is a fully indecomposable matrix D. Also the dimension of the polytope was found by Brualdi etc.

Theorem 2.1. [13] *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *be monotone with* $\mathbf{y} \prec \mathbf{x}$. *Then there exists a positive matrix in* $\Omega_n(\mathbf{y} \prec \mathbf{x})$ *if and only if one of the following holds :*

(1) $\mathbf{y}$ *is a scalar vector ;*
(2) *for* $k = 1, \ldots, n-1$, $\mathbf{y} \prec \mathbf{x}$ *does not have a coincidence at* k.

Theorem 2.2. [13] *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *be monotone with* $\mathbf{y} \prec \mathbf{x}$, *and suppose* $\mathbf{y} \prec \mathbf{x}$ *is not* k*-decomposable for each* $k = 1, \ldots, n-1$. *Then the support matrix of* $\mathbf{y} \prec \mathbf{x}$ *is the matrix* $U = [u_{ij}]$ *of 0's and 1's satisfying*

$$For \;\; 1 \le r\,, s \le n, \quad u_{rs} = 1 \;\; if\ and\ only\ if\ for\ some\ i = 1, \ldots, p,$$

$$k'_{i-1} + 1 \le r \le k_i, \quad j_{i-1} \le s \le l_i$$

or

$$k_i + 1 \le r \le k'_i, \quad j_i \le s \le l_i.$$

Theorem 2.3. [13] *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *with* $\mathbf{y} \prec \mathbf{x}$. *Then there exists a fully indecomposable matrix in* $\Omega_n(\mathbf{y} \prec \mathbf{x})$ *if and only if* $\mathbf{y} \prec \mathbf{x}$ *is not* k*-decomposable for any* $k = 1, \ldots, n-1$.

Theorem 2.4. [13] *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *be monotone with* $\mathbf{y} \prec \mathbf{x}$, *and suppose that* $\mathbf{y} \prec \mathbf{x}$ *is not* k*-decomposable for any* $k = 1, \ldots, n-1$. *Then*

$$\dim\Omega_n(\mathbf{y} \prec \mathbf{x}) = \sigma(\mathbf{y} \prec \mathbf{x}) - 3n + c(\mathbf{y} \prec \mathbf{x}) + 1.$$

Also an interest was given to the multivariate majorization polytope by Cheon and Lee in 1995. And Cheon and Lee gave a necessary and sufficient condition for $\Omega_{mn}(A \prec^{mul} B)$ to be a set of a single matrix, and they determined when $\Omega_{mn}(A \prec^{mul} B)$ contains a positive matrix. They also characterized the support matrix of the multivariate majorization $A \prec^{mul} B$ in [15].

Theorem 2.5. [15] *For $A,\ B \in M_{mn}(\mathbb{R})$, let $\mathbf{a}_i^R \prec \mathbf{b}_i^R$, $i = 1, \dots, m$. Then $A \prec^{mul} B$ if $\mathbf{a}_i^R$ and $\mathbf{b}_i^R$ are conformally partitioned as $(\mathbf{a}_i^{(p_i)}, \dots, \mathbf{a}_i^{(p_s)})$ and $(\mathbf{b}_i^{(p_i)}, \dots, \mathbf{b}_i^{(p_s)})$ respectively, where $\mathbf{a}_i^{(p_j)} \prec \mathbf{b}_i^{(p_j)}$ and $\mathbf{a}_i^{(p_j)}$ is a scalar vector with length p_j for each $j = 1, \dots, s$.*

For $1 \times n$ matrix $\mathbf{e}_n = [1, \dots, 1]$ and $A,\ B \in M_{mn}(\mathbb{R})$, let $\widehat{A} = \begin{bmatrix} \mathbf{e}_n \\ A \end{bmatrix}$ and $\widehat{B} = \begin{bmatrix} \mathbf{e}_n \\ B \end{bmatrix}$. And for rank$B = k \le n-2$, we may assume that

$$\widehat{A'} = \begin{bmatrix} 1 & \cdots & 1 \\ a'_{11} & \cdots & a'_{1n} \\ \vdots & \cdots & \vdots \\ a'_{k1} & \cdots & a'_{kn} \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad \widehat{\mathrm{B}'} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & \mathbf{b'}_1 & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & \mathbf{b'}_k \\ 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

Theorem 2.6. [15] *Let $A,\ B \in M_{mn}(\mathbb{R})$ with $A \prec B$ such that $\mathbf{a}_i^R \prec \mathbf{b}_i^R$ has no coincidences for all $i = 1, \dots, m$, let a'_{ij} and $\mathbf{b'}_i$ be as the above. Then there exists a positive matrix in $\Omega_{mn}(A \prec^{mul} B)$ if and only if one of the followings holds :*

(1) *$\mathbf{a}_i^R$ is a scalar vector for all $i = 1, \dots, m$;*
(2) *for all $i = 1, \dots, k$ and $j = 1, \dots, n$ $a'_{ij} \neq 0$ and there is no a'_{ij} such that $\mathbf{b'}_i = a'_{ij}\mathbf{e}$, where $\mathbf{e}$ is a vector of 1's of suitable sizes.*

Let $X = [x_{ij}]$ be a matrix of positive variables, and let $Y = [y_{ij}] = (D_1 * \cdots * D_m) * X$ where $*$ stands for the Hadamard product. And let $(\mathbf{y}_j{}^C)^T$ is the jth column vector of Y. Then we may assume that $\widehat{\mathbf{b'}}_i = [0 \cdots 0\ b'_{iu_i} \cdots b'_{in}]$ for $2 \le u_i \le n$ and $(\mathbf{y}_j{}^C)^T = [0 \cdots 0\ y_{s_j j} \cdots y_{l_j j}\ 0 \cdots 0]^T$ for $2 \le s_j \le l_j \le n$.

Theorem 2.7. [15] *Let D be the support matrix of $A \prec^{mul} B$ and let D_i be the support matrix of $\mathbf{a}_i^R \prec \mathbf{b}_i^R$, $i = 1, \dots, m$. Then*

$$D = D_1 * \cdots * D_m$$

if and only if both of the following conditions holds :

(1) *$a'_{ij} \neq 0$ for all (i, l) in $\mathcal{S}$;*
(2) *for all $(i, j) \in \mathcal{P}$ such that $u_i > s_j$ in the above, not all overlapping nonzero elements of $\widehat{\mathbf{b'}}_i$ and $(\mathbf{y}_j{}^C)^T$ are equal to a'_{ij}.*

This generalize the classical notion of vector majorization. Several properties and characterizations of matrix majorization were given by Dahl in [16], 1999. Moreover, interpretations of concept in mathematical statistics were discussed and some combinatorial questions studied there too.

Theorem 2.8. [16] *For each $A \in M_{mn}(\mathbb{R})$, $B \in M_{mp}(\mathbb{R})$, $C \in M_{mq}(\mathbb{R})$, the following statements hold.*

(1) *$A \succ A$.*
(2) *If $A \succ B$ and $B \succ C$, then $A \succ C$.*
(3) *If $A \succ B$, then $A[I] \succ B[I]$ for each $I \subseteq \{1, \dots, m\}$.*
(4) *If $A \succ B$ and $H \in M_m(\mathbb{R})$, then $HA \succ HB$.*
(5) *If $A \succ B$ and $P \in M_n(\mathbb{R})$ and $Q \in M_p(\mathbb{R})$ are two permutation matrices, then $AP \succ BQ$.*
(6) *If $A \succ B$, then* $\text{cone}(A) \supseteq \text{cone}(B)$ *and* $\text{rank}(A) \geq \text{rank}(B)$.
(7) *Assume that $m = n$ and that A is nonsingular. Then $A \succ B$ if and only if* $\text{span}(A) \supseteq \text{span}(B)$*, and in that case the majorization polytope is given by $\mathcal{M}_{n,p}(A, B) = \{A^{-1}B\}$.*
(8) *For each $k \geq 1$ the set $\{Z \in M_{mk}(\mathbb{R}) \;:\; Z \prec A\}$ is a polyhedron, in particular, it is convex.*
(9) *Let A' be a matrix obtained by augmenting A with some columns of all zeros. Then $A \succ A' \succ A$.*

Theorem 2.9. [16] *Let $A = [a_{ij}] \in M_{mn}(\mathbb{R})$ and $B = [b_{ij}] \in M_{mp}(\mathbb{R})$. Then the following five statements are equivalent.*
(1) *$A \succ B$.*
(2) *$\mathcal{M}(A; k) \supseteq \mathcal{M}(B; k)$ for each positive integer k.*
(3) *For each $k \geq 1$ and $L \in M_{mk}(\mathbb{R})$ we have that*

$$\min\{< A', L >:\ A' \in \mathcal{M}(A; k)\} \leq \min\{< B', L >:\ B' \in \mathcal{M}(B; k)\}.$$

(4) *For each $k \leq 1$, $L \in M_{mk}(\mathbb{R})$ and $N \in \mathcal{M}_{p,k}$ there is an $M \in \mathcal{M}_{n,k}$ such that $(AM) \diamond L \leq (BN) \diamond L$.*
(5) *For each $k \geq 1$ and $\phi \in \Phi_k(M_m(\mathbb{R}))$ we have that*

$$\sum_{j=1}^{n} \phi(\mathbf{a}^j) \geq \sum_{j=1}^{p} \phi(\mathbf{b}^j).$$

Theorem 2.10. [17] *Let $A,\ B \in M_{mn}(\mathbb{R})$ be $(0,1)$-matrices with exactly two ones in each row. Then $A \succ B$ if and only if the following conditions hold :*
(1) *If the edges f_k^B and f_l^B are disjoint (where $k,\ j \leq m$), then f_k^A and f_l^A are disjoint.*
(2) *Each nonvertical, connected component of $H_{A,B}$ is bipartite and all its cross -nodes belong to the same color class.*

Also Hwang [21] gave an equivalent definition of vector majorization in terms of generalized Hessenberg matrices in 1999. And the research on this subject still goes strong in applied science including economics and biological mathematics.

3. Linear preservers of majorizations

Over the last century, a great deal of effort has been devoted to the following problem. Let $\mathcal{A}$ be a linear space of matrices. Characterize those linear operators $T : \mathcal{A} \to \mathcal{A}$ which leave a function or set invariant. We call this *a linear preserver problem.* Linear preserver problems usually fall into one of four categories.

(i) Suppose that $\mathcal{P}$ is a certain property of matrices. Characterize those linear operator T on $\mathcal{A}$ that preserve property $\mathcal{P}$ in the sense that

$$T(X) \ satisfies\ \mathcal{P}\ whenever\ X\ satisfies\ \mathcal{P}.$$

For example, let $\mathcal{A}$ be the set of all $n \times n$ complex matrices, and let $\mathcal{P}$ be nonsingularity. Then the problem is to classify all linear maps T satisfy

$$T(X) \ \textit{is nonsingular whenever} \ X \ \textit{is nonsingular}.$$

The answer was given in [26]. The map T preserves nonsingularity if and only if there exist invertible $n \times n$ complex matrices U and V such that

$$T(X) = UXV \ \ \textit{for all} \ \ X \in \mathcal{A}.$$

or

$$T(X) = UX^TV \ \ \textit{for all} \ \ X \in \mathcal{A}$$

where X^T denotes the transpose of X. It should be noted that many linear preserver problems have answers like (or nearly like) above, where U and V have to meet certain specifications. Those operators T are called to be (U, V)-*operators.*

(ii) Suppose that $\mathcal{S}$ is a subset of $\mathcal{A}$. Characterize those linear operators T on $\mathcal{A}$ that map $\mathcal{S}$ into itself.

For example, let $\mathcal{A}$ be the $n \times n$ complex matrices and let $\mathcal{S}$ be the group of *unitary matrices.* Then (see [25]) the necessary and sufficient condition for a linear transformation T preserves the *unitary group* is that T is a (U, V)-operator with U and V unitary.

(iii) Let F be a scalar-valued, vector-valued, or set-valued function on $\mathcal{A}$. Characterize those linear operators T on $\mathcal{A}$ that preserve F in the sense that $F(T(X)) = F(X)$ for all $X \in \mathcal{A}$.

For example, let $\mathcal{A}$ be the set of $n \times n$ symmetric matrices over any subfield of the real field with $n \geq 3$ and $F(X)$ be the *permanent* of X. Then a linear operator T on $\mathcal{A}$ preserves permanent if and only if

$$T(A) = PAP^T$$

for a fixed *generalized permutation matrix* P with $F(P) = 1$ in [23].

(iv) Let $\mathcal{R}$ be a certain relation on $\mathcal{A}$. Characterize those linear operators on $\mathcal{A}$ that preserve $\mathcal{R}$ in the sense that

$$\mathcal{R}(T(X), T(Y)) \ \ \textit{whenever} \ \ \mathcal{R}(X, Y).$$

For example, let $\mathcal{A}$ be the $n \times n$ complex matrices. If T is nonsingular and $XY = YX$ implies $T(X)T(Y) = T(Y)T(X)$, then there exist a nonsingular matrix $S \in \mathcal{A}$, a scalar $c \in \mathcal{F}$, and a linear functional f on $\mathcal{A}$, such that either

$$T(X) = cS^{-1}XS + f(X)I \ \ \textit{for all} \ \ X \in \mathcal{A}$$

or

$$T(X) = cS^{-1}X^TS + f(X)I \ \ \textit{for all} \ \ X \in \mathcal{A}$$

(see [4], [22], and [29]).

One may also consider the stronger statement

$$\mathcal{R}(T(X), T(Y)) \ \ \textit{if and only if} \ \ \mathcal{R}(X, Y).$$

for $\mathcal{A}$ be a linear space of matrices, T be a linear operator on $\mathcal{A}$ and $\mathcal{R}$ be a relation on $\mathcal{A}$. We say a linear operator T *strongly preserves* $\mathcal{R}$ if T satisfies the above statement.

The following is the characterization of vector majorization preserves which was given by Ando in 1989.

Theorem 3.1. [2] *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ preserve (vector) majorization, then either*

(1) $T(\mathbf{x}) = (\sum_{i=1}^n x_i)\mathbf{a}$ *for some* $\mathbf{a} \in \mathbb{R}^n$ *or*
(2) $T(\mathbf{x}) = \beta \mathbf{x} P + \gamma(\sum_{i=1}^n x_i)\mathbf{j}$ *for some* $\beta,\ \gamma \in R^n$ *and* $P \in \mathbf{P}(n)$.

The following is the characterization of majorization preservers on $H_n(\mathbb{C})$ which was given by Hiai in 1987.

Theorem 3.2. [20] *For each linear map Φ on $H_n(\mathbb{C})$, the following conditions are equivalent :*

(1) *Φ preserves majorization ;*
(2) *$A \prec B$ implies $\Phi(A) \prec \Phi(B)$ for every $A,\ B \in H_n(\mathbb{C})$;*
(3) *Either*
   (a) *there exists an $A_0 \in H_n(\mathbb{C})$ such that $\Phi(X) = \mathrm{tr}(X)A_0$ for all $X \in H_n(\mathbb{C})$ or*
   (b) *there exist a unitary matrix U and $\alpha,\ \beta \in \mathbb{R}$ such that Φ has one of the following forms :*
      (b-1) $\Phi(X) = \alpha U X U^* + \beta \mathrm{tr}(X)I$ *for all* $X \in H_n(\mathbb{C})$
      (b-2) $\Phi(X) = \alpha U X^T U^* + \beta \mathrm{tr}(X)I$ *for all* $X \in H_n(\mathbb{C})$

In 1999, Beasley, Lee and Lee extended their results for the matrix version.

Theorem 3.3. [6] *If $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ strongly preserves multivariate majorization, $T(I) = I$, and $T : M_n(\mathbb{R}^+) \to M_n(\mathbb{R}^+)$ (i.e., T preserves nonnegative matrices). Then there exists a permutation matrix P such that $T(X) = P^T X P$ for every $X \in M_n(\mathbb{R})$.*

And they conjectured the condition of $T(I) = I$ can be removed from the main theorem in the paper. They also recently resolved the conjecture. The following theorem gave the characterization of linear preserves without the condition $T(I) = I$.

Theorem 3.4. [7] *Let $T : M_n(\mathbb{R}^+) \to M_n(\mathbb{R}^+)$ be a linear operator that strongly preserves multivariate majorization, then there exists a invertible matrix C and $P \in \mathbf{P}(n)$ such that $T(A) = CAP$ for every $A \in M_n(\mathbb{R}^+)$.*

In 2000, Beasley and Lee extended the results on multivariate majorization preservers as following.

Theorem 3.5. [5] *Let T be a linear operator on $M_{mn}(\mathbb{R})$. Then the followings are equivalent :*

(1) *T preservers multivariate majorization.*
(2) *Either*
   (a) *there exist matrices $G \in M_m(0,1)$ and $H_1, \ldots, H_m \in M_{mn}(\mathbb{R})$ such that $T(X) = \sum_{l=1}^m D_l G D(X) H_l$ for all $X \in M_{mn}(\mathbb{R})$ or*
   (b) *there exist matrices $B,\ C \in M_{mn}(\mathbb{R})$ and a permutation matrix $P \in \mathbf{P}(n)$ such that $T(X) = BXP + CD(X)\mathbf{J}$ for all $X \in M_{mn}(\mathbb{R})$.*

This concluded the previous multivariate majorization preserver problems.

Recently C. K. Li and E. Poon gave a characterization for the preservers of directional majorization. A very interesting fact is the above two characterization was equivalent. It was shown in the 2000 Fall meeting of Korean Math Society.

Beside the above results, since Dahl gave several results for the characterization of matrix majorization which generalize the multivariate majorization in 1999, it opened a sequence of matrix majorization problems. So it is natural to give the characterization of the preservers of it. The following is the recent result of Lee and Lee on the strong preservers of matrix majorization.

Theorem 3.6. [8] *Let $T : M_n(\mathbb{R}^+) \to M_n(\mathbb{R}^+)$ be a linear operator that strongly preserves matrix majorization. If T preserves nonnegative matrices, then, for some invertible $M \in M_n(\mathbb{R}^+)$ and $P \in \mathbf{P}(n)$,*

$$T(A) = MAP.$$

Now we have something to work with matrix majorization and directional majorization.

4. Conclusion

We conclude this survey by starting some open problems.

Problems :

What is the complete characterization of matrix majorization preservers without any condition?

What are the characterization for the matrix majorization version of various concepts?

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