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A REALIZATION OF SYMMETRIC GEODESICS DERIVED FROM Sym$(2,\mathbb{R})$ AS KNOTS IN $S^2 \times S^1$

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Abstract. The purpose of this talk is to report some recent results, joint-work with S. Y. Lee and Y. Lim, (i) on the classification of closed geodesics and symmetric geodesics on the real conformal compactification of the space Sym(n, $\mathbb{R}$) of all $n \times n$ real symmetric matrices([8]), (ii) on the realization of closed geodesics on the real conformal compactification of the space Sym$(2,\mathbb{R})$ as knots or 2-component links in $S^2 \times S^1$([9]) and (iii) on the classification of these knots or links as certain types of symmetries of period 2([9]).

1. Introduction

A commutative algebra V over a field $\mathbb{R}$ or $\mathbb{C}$ with product xy is said to be a *Jordan algebra* if for all elements x, y in V, $x(x^2y) = x^2(xy)$. For $x \in V$, let $L(x)$ be the linear map of V defined by $L(x)y = xy$, and let $P(x) = 2L(x)^2 - L(x^2)$. A finite dimensional real Jordan algebra V is called a *Euclidean Jordan algebra* if it admits an inner product $\langle x|y\rangle$ such that $\langle xy|\ z\rangle = \langle y|\ xz\rangle$. Let V be a Euclidean Jordan algebra with identity e and let Q be the set of squares, and let Ω be the interior of Q. Then Ω is a self-dual cone and the group $G(\Omega) := \{g \in GL(V) \mid g\Omega = \Omega\}$ acts on it transitively. The tube domain $T_\Omega := V + i\Omega$ associated with Ω is a symmetric tube domain which is biholomorphically isomorphic to a bounded symmetric domain via a Cayley transform. The Lie group $G(T_\Omega)$ of all biholomorphic automorphisms on the tube domain T_Ω can be described in the following way:

An element in $G(\Omega)$ acts on the tube domain T_Ω by $g(z) = g(x)+ig(y), z = x+iy$. For $x \in V$, the translation by x, $t_x : z \longrightarrow z + x$ is a holomorphic automorphism of T_Ω and the group N^+ of all real translations is an abelian group isomorphic to the vector space V. The map $j : z \longrightarrow -z^{-1}$, the symmetry at ie, belongs to $G(T_\Omega)$. Let $\tilde{t_x} = j \circ t_x \circ j$, $N^- = j \circ N^+ \circ j$. Then $G(T_\Omega)$ is generated by $N^+, G(\Omega)$ and j. Let $\mathfrak{n}^\pm$ be the Lie algebra of $N^\pm$ and let $\mathfrak{h}$ be the Lie algebra of $G(\Omega)$. Then the Lie algebra $\mathfrak{g}(T_\Omega)$ of $G(T_\Omega)$ can be identified with $\mathfrak{n}^+ +\ \mathfrak{h} + \mathfrak{n}^-$ ([3]).

It is known in ([7]) that $\mathfrak{g}(T_\Omega)$ is a symmetric algebra of Cayley type. Hence

$$(X, h, Y) \in \mathfrak{n}^+ \times G(\Omega) \times \mathfrak{n}^- \longrightarrow (\exp X)h(\exp Y) \in N^+G(\Omega)N^-$$

is a diffeomorphism and $N^+G(\Omega)N^-$ is a dense open subset in $G(T_\Omega)$. Set $P = G(\Omega)N^-$. Then P is a maximal parabolic subgroup of $G(T_\Omega)$ and the homogeneous space $\mathcal{M} := G(T_\Omega)/P$ is a compact real manifold containing V as an open dense

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subset, i.e., a real *conformal compactification* of the Jordan algebra V. The embedding $V \longrightarrow \mathcal{M}$ is given by $x \to t_x P$. Furthermore, the set $N^+G(\Omega)N^-$ can be characterized by the elements $g \in G(T_\Omega)$ such that $g \cdot \mathbf{0} \in V$, where $\mathbf{0}$ is the base point P corresponding to the zero vector in V ([2], [5]).

To complete a semisimple Jordan algebra V of classical type to a symmetric space, B. Makarevich ([10]) used the term of geodesics in V that originate at the zero point. In the case of the Euclidean (or formally real) Jordan algebra $V = \mathrm{Sym}(n, \mathbb{R})$ of all $n \times n$ real symmetric matrices, these geodesics are eventually of the form $\alpha(t, A) := \exp tX_A \cdot \mathbf{0}$, $A \in V$, where $X_A = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} \in \mathfrak{sp}(2n, \mathbb{R})$, the Lie algebra of the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$. In [8], the authors classified closed geodesics and symmetric geodesics of these types on the real conformal compactification $\mathcal{M}$ of $V = \mathrm{Sym}(n, \mathbb{R})$. In section 2, we give a brief review of these results together with some elementary facts, for our convenience.

It is known in ([1]) that the conformal compactification $\mathcal{M}$ of $V = \mathrm{Sym}(n, \mathbb{R})$ is diffeomorphic to the Shilov boundary Σ_n of the symmetric tube domain $T_\Omega = V + i\Omega$, where Ω is the open convex cone of all positive definite $n \times n$ symmetric matrices. In ([9]), the authors gave a realization of these closed geodesics in the Shilov boundary Σ_2 as knots in $S^2 \times S^1$ and characterizes their symmetric properties. we focus on these results in this talk but we omitt all the proofs here. Please see ([8], [9]) for the complete proofs or contact me via the e-mail address at the end.

Throughout this paper, all maps and spaces will be assumed to be in the piecewise-linear(PL) category. A *link* L of μ-components in a connected 3-manifold M is (or the image of) an embedding of μ disjoint union of 1-spheres into M. If $\mu = 1$, then L is called a *knot* in M. Two links L and L' are said to be *equivalent* if there exists an ambient isotopy $H : M \times [0,1] \to M \times [0,1]$, $H(x,t) = h_t(x)(t \in [0,1])$, such that h_0 is the identity on M and $h_1(L) = L'$.

A knot(or link) K in a connected 3-manifold M is said to have *period n of type* (X, Y)(or an *n-periodic knot of type* (X, Y)) if there is an n-periodic homeomorphism $h : (M, K) \to (M, K)$ such that the fixed point set, $\mathrm{Fix}(h)$, of h is homeomorphic to X and $\mathrm{Fix}(h) \cap L$ is homeomorphic to Y. If M is a homology 3-sphere, then P. A. Smith([12]) proved that the set of all fixed points of a periodic homeomorphism of M is $\emptyset, S^0, S^1$, or S^2. By the positive solution of the Smith conjecture([11]), the possible types for non trivial knots in the 3-sphere S^3 are $(\emptyset, \emptyset), (S^0, \emptyset), (S^0, S^0), (S^1, \emptyset)$, (S^1, S^0), and (S^2, S^0)([4]).

In section 3, we show that the closed geodesics on conformal compactification $\mathcal{M}$ of $\mathrm{Sym}(2, \mathbb{R})$ are knots in the Shilov boundary Σ_2 which have period 2 of type both $(\emptyset, \emptyset)$ and $(S^1 \cup S^0, S^0)$.

In ([6], [13], [14]), it was shown that $S^2 \times S^1$ admits exactly thirteen distinct involutions (up to conjugation) and the possible types of the fixed point sets for the involutions are $\emptyset, S^0 \dot\cup S^0, S^1, S^1 \dot\cup S^1, S^1 \times S^1$, Klein bottle, $S^0 \dot\cup S^2$, or $S^2 \dot\cup S^2$, where $X \dot\cup Y$ denotes the disjoint union of X and Y.

In section 4, we give an explicit description of an orientable double cover $S^2 \times S^1$ of the Shilov boundary Σ_2 and show that the knots in Σ_2 corresponding to the closed geodesics are lifted to knots or links of 2-components in $S^2 \times S^1$ and show that these knots or links in $S^2 \times S^1$ have also period 2 of types $(S^1 \times S^1, T(m,n)), (S^1 \times S^1, S^0 \dot\cup \cdots \dot\cup S^0)$, or $(S^1 \dot\cup S^1, \emptyset)$, where $T(m,n)$ denotes the torus knot of type (m,n).

2. GEODESICS ON THE CONFORMAL COMPACTIFICATION OF $\mathrm{Sym}(n, \mathbb{R})$

Let $M_n(\mathbb{R})$ denote the space of all $n \times n$ real matrices. A symmetric (respectively, skew-symmetric) matrix $A \in M_n(\mathbb{R})$ means that $A^t = A$ (respectively, $A^t = -A$), where A^t denotes the transpose of a matrix A. Let $\mathrm{Sym}(n, \mathbb{R})$ (respectively, $\mathrm{Skew}(n, \mathbb{R})$) be the space of all symmetric (respectively, skew-symmetric) $n \times n$ matrices. Let $A \in \mathrm{Sym}(n, \mathbb{R})$ have the spectral decomposition $A = \sum_{k=1}^{n} \lambda_k C_k$, where $\{C_k\}$ is a complete system of orthogonal projections. Then the spectral norm $|A|$ of A is defined by $|A| = \max\{|\lambda_1|, \cdots, |\lambda_n|\}$.

Let $(\cdot|\cdot)$ be the skew-symmetric form on $\mathbb{R}^{2n}$ defined by $(u|v) = \langle Ju \mid v\rangle$ for $u, v \in \mathbb{R}^{2n}$, where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Here, I stands for the $n \times n$ identity matrix. The *symplectic group* $\mathrm{Sp}(2n, \mathbb{R})$ on $\mathbb{R}^{2n}$ is the Lie group of all invertible transformations $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ satisfying one of the following equivalent conditions:

(1) g preserves $(\cdot|\cdot)$.
(2) $g^t J g = J$.
(3) $A^t C, B^t D$ are symmetric and $A^t D - C^t B = I$.

The Lie algebra of $\mathrm{Sp}(2n, \mathbb{R})$ is given by

$$\mathfrak{sp}(2n, \mathbb{R}) = \left\{ \begin{pmatrix} X & Y \\ Z & -X^t \end{pmatrix} \mid X \in M_n(\mathbb{R}), Y, Z \in \mathrm{Sym}(n, \mathbb{R}) \right\}.$$

It has a Cartan decomposition $\mathfrak{sp}(2n, \mathbb{R}) = \mathfrak{p} \oplus \mathfrak{k}$, where

$$\begin{aligned} \mathfrak{p} &= \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \mid X, Y \in \mathrm{Sym}(n, \mathbb{R}) \right\}, \\ \mathfrak{k} &= \left\{ \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \mid X \in \mathrm{Skew}(n, \mathbb{R}), Y \in \mathrm{Sym}(n, \mathbb{R}) \right\}. \end{aligned}$$

Let $\tau = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \in GL(2n, \mathbb{R})$ and let $\tau(g) = \tau \cdot g \cdot \tau$ for $g \in \mathrm{Sp}(2n, \mathbb{R})$. Then τ is an involution on $\mathrm{Sp}(2n, \mathbb{R})$. The differential $\mathrm{d}\tau$ of τ at the identity is given by

$$\mathrm{d}\tau \begin{pmatrix} X & Y \\ Z & -X^t \end{pmatrix} = \begin{pmatrix} X & -Y \\ -Z & -X^t \end{pmatrix}.$$

The Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$ can be decomposed as a direct sum of the $(+1)$-eigenspace $\mathfrak{h}$ and the (-1)-eigenspace $\mathfrak{q}$ of $\mathrm{d}\tau$:

$$\mathfrak{sp}(2n, \mathbb{R}) = \mathfrak{h} \oplus \mathfrak{q} = \mathfrak{h} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-, \quad \mathfrak{q} = \mathfrak{n}^+ \oplus \mathfrak{n}^-,$$

where

$$\begin{aligned} \mathfrak{n}^+ &= \left\{ \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} \mid Y \in \mathrm{Sym}(n, \mathbb{R}) \right\}, \\ \mathfrak{n}^- &= \left\{ \begin{pmatrix} 0 & 0 \\ Z & 0 \end{pmatrix} \mid Z \in \mathrm{Sym}(n, \mathbb{R}) \right\}, \\ \mathfrak{h} &= \left\{ \begin{pmatrix} X & 0 \\ 0 & -X^t \end{pmatrix} \mid X \in M_n(\mathbb{R}) \right\}. \end{aligned}$$

Let $N^{\pm}$ be the Lie subgroups of $\mathrm{Sp}(2n,\mathbb{R})$ corresponding to $\mathfrak{n}^{\pm}$ respectively. Then

$$\begin{aligned} N^{+} &= \left\{ \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \mid A \in \mathrm{Sym}(n,\mathbb{R}) \right\} = \exp \mathfrak{n}^{+}, \\ N^{-} &= \left\{ \begin{pmatrix} I & 0 \\ A & I \end{pmatrix} \mid A \in \mathrm{Sym}(n,\mathbb{R}) \right\} = \exp \mathfrak{n}^{-}. \end{aligned}$$

Let $H = \{ g \in \mathrm{Sp}(2n,\mathbb{R}) \mid \tau(g) = g \}$. We observe that

$$H = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^t \end{pmatrix} \mid A \in GL(n,\mathbb{R}) \right\}.$$

Theorem 2.1. [1] *Let $P = HN^{-}$. Then P is a closed subgroup of $G := \mathrm{Sp}(2n,\mathbb{R})$ and the homogeneous space $\mathcal{M} := G/P$ is a compact real manifold with $V := \mathrm{Sym}(n,\mathbb{R})$ as an open dense subset. The embedding of V into $\mathcal{M}$ is given by*

$$X \in V \longrightarrow \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \cdot P \in \mathcal{M}.$$

Furthermore, for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n,\mathbb{R})$ and $X \in V$ with $g \cdot X \in V$, we have $g \cdot X = (AX+B)(CX+D)^{-1}$.

Let $A \in V = \mathrm{Sym}(n,\mathbb{R})$ and let $X_A := \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} \in \mathfrak{k}$. Then it is known in ([10]) that the geodesic in $\mathcal{M}$ originating at the origin $\mathbf{0}$ with direction A is of the following form

$$\alpha(t,A) := \exp tX_A \cdot \mathbf{0} = \begin{pmatrix} \cos tA & \sin tA \\ -\sin tA & \cos tA \end{pmatrix} \cdot \mathbf{0}.$$

The *period* of a non-constant closed geodesic $\alpha(t,A)$ is the smallest positive real number t_0 satisfying $\alpha(t_0,A) = \mathbf{0}$.

Set $j = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \in \mathrm{Sp}(2n,\mathbb{R})$. Then j is an involution on $\mathcal{M}$ and for an invertible element $A \in V$, $j \cdot A = -A^{-1}$. A closed geodesic $\alpha(t,A)$ is said to be *symmetric* if it is invariant under the involution j on $\mathcal{M}$.

Let $E_c = \{ r(p_1,\cdots,p_n) \in \mathbb{R}^n \mid r \geq 0, p_i : \text{integers} \}$, $E_s = \{ r(p_1,p_2,\cdots,p_n) \in E_c \mid r > 0, p_i : \text{odd integers} \}$. In this setting, we always assume that all integers p_i have no common divisors. Let $A = \sum_{k=1}^{n} \lambda_k C_k$ be the spectral decomposition of A. Then $\alpha(t,A)$ is a closed geodesic if and only if $(\lambda_1,\cdots,\lambda_n) \in E_c$. If $A \neq 0$ and $(\lambda_1,\cdots,\lambda_n) = r(p_1,\cdots,p_n) \in E_c$, then $\frac{\pi}{r}$ is the period of $\alpha(t,A)$ ([8]), Theorem 4.2). Also, $\alpha(t,A)$ is a symmetric geodesic if and only if $(\lambda_1,\cdots,\lambda_n) \in E_s$ ([8]), Theorem 4.4).

Now let Ω be the symmetric cone of positive definite $n \times n$ symmetric real matrices. Then the tube domain $T_\Omega := V + i\Omega$ can be realized as a bounded symmetric domain $\mathcal{D}$ in the complex plane $V^{\mathbb{C}} := V + iV$ as follows: Let

$$\begin{aligned} D(p) &= \{ Z \in V^{\mathbb{C}} \mid Z + iI \in GL(n,\mathbb{C}) \}, \\ D(c) &= \{ W \in V^{\mathbb{C}} \mid I - W \in GL(n,\mathbb{C}) \}, \end{aligned}$$

and define for all $Z \in D(p)$ and $W \in D(c)$,

$$\begin{aligned} p(Z) &= (Z - iI)(Z + iI)^{-1} \\ c(W) &= i(I + W)(I - W)^{-1}. \end{aligned}$$

Then the map $p : D(p) \to D(c)$ is a holomorphic bijection from $D(p)$ onto $D(c)$ and the map $c : D(c) \to D(p)$, called the *Cayley transform*, is its inverse. The closure of T_Ω in $V^{\mathbb{C}}$ is contained in $D(p)$. The image $\mathcal{D} := p(T_\Omega)$ of p is known as a bounded symmetric domain which is the open unit ball with respect to the spectral norm. We define Σ_n as the set of all invertible elements Z in $V^{\mathbb{C}}$ such that $Z^{-1} = \overline{Z}$. It is known that Σ_n is the Shilov boundary of $\mathcal{D}$, which is a compact connected $\frac{n(n+1)}{2}$-dimensional manifold, and is exactly equal to $\overline{p(V)}$ (For details, see [3]).

Let $\mathbf{c} = \{C_k\}_{k=1}^n$ be a complete system of orthogonal projections and let $V(\mathbf{c})$ be the subspace of V generated by C_k's. Then for $A = \sum_{k=1}^n \lambda_k C_k$,

$$p(A) = \sum_{k=1}^{n} \frac{\lambda_k - i}{\lambda_k + i} C_k. \tag{2.1}$$

Since $\frac{\lambda_k - i}{\lambda_k + i} \in S^1$ (the unit circle in $\mathbb{C}$) for $k = 1, 2, \cdots, n$, we conclude that $\overline{p(V(\mathbf{c}))}$ is diffeomorphic to the n-torus $T^n = S^1 \times S^1 \times \cdots \times S^1$.

For a geodesic curve $\alpha(t, A)$ on $\mathcal{M}$, we let $\hat{\alpha}(t, A) := p(\alpha(t, A))$ be the corresponding geodesic on Σ_n. From (2.1), we have the following

Proposition 2.2. [9] *Let $A = \sum_{k=1}^n \lambda_k C_k$ be the spectral decomposition of A. Then*

$$\hat{\alpha}(t, A) = \sum_{k=1}^{n} e^{i(\pi + 2\lambda_k t)} C_k.$$

The symmetry $\hat{j} := p \circ j \circ c$ on Σ_n corresponding to the symmetry j on $\mathcal{M}$ is the symmetry about the origin, i.e., $\hat{j}(Z) = -Z$. Let J be the involution on Σ_n defined by $J(Z) = -\overline{Z}$. Then since $\overline{Z} = Z^{-1}$ for any $Z \in \Sigma_n$, this involution is just $J(Z) = -Z^{-1}$ and $J = j$ on $V = \mathrm{Sym}(n, \mathbb{R})$. By Proposition 2.2, we have

Corollary 2.3. [9] *If $\alpha(t, A)$ is a symmetric geodesic on $\mathcal{M}$ with $A = \sum_{k=1}^n r p_k C_k$ then $\hat{\alpha}(t, A)$ is invariant under the involutions both $\hat{j}$ and J on Σ_n.*

3. Closed geodesics in Shilov boundary Σ_2

From now on, we shall restrict our attention to the space $V = \mathrm{Sym}(2, \mathbb{R})$. Recall that the Shilov boundary Σ_2 of V is given by

$$\Sigma_2 = \{Z \in \mathrm{Sym}(2, \mathbb{C}) \mid Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \text{ is an invertible matrix with } \bar{Z} = Z^{-1}\}.$$

We identify $Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in \mathrm{Sym}(2, \mathbb{C})$ with $(z_1, z_2, z_3) \in \mathbb{C}^3$. Under this identification, the Shilov boundary Σ_2 of $\mathrm{Sym}(2, \mathbb{R})$ can be written as

$$\Sigma_2 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 \overline{z}_2 + z_2 \overline{z}_3 = 0, |z_1|^2 + |z_2|^2 = |z_2|^2 + |z_3|^2 = 1\}.$$

Let $A (\neq 0) \in \mathrm{Sym}(2, \mathbb{R})$ and let $\alpha(t, A)$ be the closed geodesic in $\mathcal{M}$ originating at the origin $\mathbf{0}$ with the direction A. Let $A = \lambda_1 C_1 + \lambda_2 C_2$ be the spectral decomposition of A. Then, from Theorem 4.2 in ([8]), we have that $\lambda_1 = rp$ and $\lambda_2 = rq$ for some real number $r > 0$ and coprime integers p and q. Set $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

and $E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Note that $\mathbf{e} = \{E_1, E_2\}$ is a complete system of orthogonal projections. It is well-known that the orthogonal group SO(2) acts transitively on the set of complete systems of orthogonal projections. Thus there exists a unique orthogonal matrix $P_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}(2)(\theta \in [0,\pi])$ such that $C_i = P_\theta E_i P_\theta^t (i = 1,2)$, i.e., $A = P_\theta A_0 P_\theta^t$, where $A_0 := (rp)E_1 + (rq)E_2$. By Proposition 2.2 and Lemma 2.4, we obtain that $\hat{\alpha}(t, A) =$

$$\begin{pmatrix} \cos^2\theta e^{i(\pi+2rpt)} + \sin^2\theta e^{i(\pi+2rqt)} & \sin\theta\cos\theta(e^{i(\pi+2rqt)} - e^{i(\pi+2rpt)}) \\ \sin\theta\cos\theta(e^{i(\pi+2rqt)} - e^{i(\pi+2rpt)}) & \sin^2\theta e^{i(\pi+2rpt)} + \cos^2\theta e^{i(\pi+2rqt)} \end{pmatrix}.$$

Definition 3.1. Let $A(\neq 0) \in \mathrm{Sym}(2,\mathbb{R})$ be a 2×2 symmetric matrix which has the spectral decomposition $A = (rp)C_1 + (rq)C_2$, where $r > 0$ is a real number and p and q are coprime integers, and let $\alpha(t, A)$ be the closed geodesic in the conformal compactification $\mathcal{M}$ of $\mathrm{Sym}(2,\mathbb{R})$ originating at the origin $\mathbf{0}$ with the direction A.

(1) $\kappa(A_0)$ is a knot in Σ_2 defined by $\kappa(A_0) = \{\hat{\alpha}(t, A_0) \in \Sigma_2 \mid 0 \le t \le \frac{\pi}{r}\}$, i.e., $\kappa(A_0) = \{(e^{i(\pi+2ps)}, 0, e^{i(\pi+2qs)}) \in \Sigma_2 \mid 0 \le s \le \pi\}$.

(2) $\kappa(A)$ is a knot in Σ_2 defined by $\kappa(A) = \{\hat{\alpha}(t, A) \in \Sigma_2 \mid 0 \le t \le \frac{\pi}{r}\}$, i.e., $\kappa(A) = \{(z_1(s), z_2(s), z_3(s)) \in \Sigma_2 \mid 0 \le s \le \pi\}$, where

$$\begin{aligned} z_1(s) &= -\cos^2\theta e^{i(2ps)} - \sin^2\theta e^{i(2qs)}, \\ z_2(s) &= \sin\theta\cos\theta(e^{i(2ps)} - e^{i(2qs)}), \\ z_3(s) &= -\sin^2\theta e^{i(2ps)} - \cos^2\theta e^{i(2qs)}, \end{aligned}$$

and $\theta \in (0,\pi)$ satisfying $C_i = P_\theta E_i P_\theta^t (i = 1,2)$.

Theorem 3.2. [9] *Let $A \in \mathrm{Sym}(2,\mathbb{R})$ be a 2×2 symmetric matrix such that $\alpha(t, A)$ is a symmetric geodesic in $\mathcal{M}$. Then the knots $\kappa(A_0)$ and $\kappa(A)$ in Σ_2 are equivalent and $\kappa(A)$ in Σ_2 has a period 2 of type both $(\emptyset, \emptyset)$ and $(S^1 \cup S^0, S^0)$.*

4. Covering links of the closed geodesics

Let $N = S^1 \times [0,1]$ be an annulus in $\mathbb{R}^3$ and let Φ be the map from $N \times S^1$ to Σ_2 defined by

$$\Phi(e^{i\phi}, r, e^{i\psi}) = (\sqrt{1-r^2}e^{i\phi}, re^{i\psi}, -\sqrt{1-r^2}e^{i(2\psi-\phi)})$$

for $(e^{i\phi}, r, e^{i\psi}) \in S^1 \times [0,1] \times S^1$. Note that $\Phi(e^{i\phi}, 0, e^{i\psi}) = (e^{i\phi}, 0, -e^{i(2\psi-\phi)})$ and $\Phi(e^{i\phi}, 1, e^{i\psi}) = (0, e^{i\psi}, 0)$. Now let $(z_1, z_2, z_3) \in \Sigma_2$ with $z_1 = re^{i\phi} \in \mathbb{C}$. Then $0 \le r \le 1$ and for some $\psi \in [0, 2\pi]$, we have the following.

(4-1) If $r = 0$, then $(z_1, z_2, z_3) = (0, e^{i\psi}, 0)$ and

$$\Phi^{-1}(z_1, z_2, z_3) = \{(e^{i\phi}, 1, e^{i\psi}) \mid 0 \le \phi \le 2\pi\}.$$

(4-2) If $r = 1$, then $(z_1, z_2, z_3) = (e^{i\phi}, 0, e^{i\psi})$ and

$$\Phi^{-1}(z_1, z_2, z_3) = (e^{i\phi}, 0, \pm e^{i(\pi+\phi+\psi)/2}).$$

(4-3) If $0 < r < 1$, then $(z_1, z_2, z_3) = (re^{i\phi}, \sqrt{1-r^2}e^{i\psi}, -re^{i(2\psi-\phi)})$ and

$$\Phi^{-1}(z_1, z_2, z_3) = (\frac{1}{|z_1|}z_1, \sqrt{1-|z_1|^2}, \frac{1}{\sqrt{1-|z_1|^2}}z_2) = (e^{i\phi}, \sqrt{1-r^2}, e^{i\psi}).$$

This shows that Σ_2 is an identification space of $N \times S^1 = S^1 \times [0,1] \times S^1$. In fact, this observation gives us the following

Theorem 4.1. [9] *The Shilov boundary* Σ_2 *of* $\mathrm{Sym}(2,\mathbb{R})$ *is homeomorphic to the nonorientable closed* 3*-manifold obtained from the solid torus* $D^2 \times S^1$ *by identifying* (w,z) *with* $(w,-z)$ *for each* (w,z) *in the boundary* $\partial D^2 \times S^1$ *of the solid torus.*

Now let $S^2 = \{(\sqrt{1-r^2}e^{i\phi}, r) \in \mathbb{C} \times \mathbb{R} \mid 0 \leq \phi \leq 2\pi, -1 \leq r \leq 1\}$ be the unit sphere in $\mathbb{R}^3$ and let $\hat{N} = S^1 \times [-1,1]$ be an annulus in $\mathbb{R}^3$. Let $f : \hat{N} \to S^2$ be a map defined by $f(e^{i\phi}, r) = (\sqrt{1-r^2}e^{i\phi}, r)$ for $(e^{i\phi}, r) \in \hat{N}$ and let $g : \hat{N} \to N$ be a map defined by $g(e^{i\phi}, r) = (e^{i\phi}, |r|)$ for $(e^{i\phi}, r) \in \hat{N}$. It is easy to see that the 2-sphere S^2 is an identification space of $\hat{N}$ and the map g is a 2-fold branched covering projection with a branch set $S^1 \times \{0\} \subset N$. The preimage of this branch set by g is $S^1 \times \{0\} \subset \hat{N}$. Let $\Psi : S^2 \times S^1 \to \Sigma_2$ be the map defined by $\Psi = \Phi \circ (g \times \mathrm{Id}_{S^1}) \circ (f \times \mathrm{Id}_{S^1})^{-1}$, where Id_{S^1} denotes the identity map from S^1 onto itself:

$$\begin{array}{ccc} \hat{N} \times S^1 & \xrightarrow{f \times Id_{S^1}} & S^2 \times S^1 \\ {\scriptstyle g \times Id_{S^1}} \downarrow & & \downarrow {\scriptstyle \Psi} \\ N \times S^1 & \xrightarrow{\Phi} & \Sigma_2 \end{array} \tag{4.2}$$

We observe that for $(\sqrt{1-r^2}e^{i\phi}, r, e^{i\psi}) \in S^2 \times S^1$,

$$\Psi(\sqrt{1-r^2}e^{i\phi}, r, e^{i\psi}) = (\sqrt{1-r^2}e^{i\phi}, |r|e^{i\psi}, -\sqrt{1-r^2}e^{i(2\psi-\phi)}).$$

Then it is not difficult to see that the map Ψ is a 2-fold covering projection and hence $S^2 \times S^1$ is an orientable double cover of the Shilov boundary Σ_2 of $\mathrm{Sym}(2,\mathbb{R})$.

Let $\hat{\kappa}(A_0) := \Psi^{-1}(\kappa(A_0))$ and $\hat{\kappa}(A) := \Psi^{-1}(\kappa(A))$. From $(4-1), (4-2), (4-3)$, and (4.2) we obtain a certain class of knots and links in $S^2 \times S^1$ as follow.

Definition 4.2. Let $A (\neq 0) \in \mathrm{Sym}(2,\mathbb{R})$ be a 2×2 symmetric matrix which has the spectral decomposition $A = (rp)C_1 + (rq)C_2$, where $r > 0$ is a real number and p and q are coprime integers.

(1) $\hat{\kappa}(A_0) = \{(e^{i(\pi+2ps)}, 0, \pm e^{i(\frac{3\pi}{2}+(p+q)s)}) \in S^2 \times S^1 \mid 0 \leq s \leq \pi\}$.
(2) $\hat{\kappa}(A) = \{a(s) \in S^2 \times S^1 \mid 0 \leq |z_1(s)| < 1, 0 \leq s \leq \pi\} \cup$
$\{\Psi^{-1}(z_1(s), 0, z_3(s)) \in S^2 \times S^1 \mid s = \frac{k\pi}{p-q}$ for $k \in \mathbb{Z}$ with $0 \leq \frac{k}{p-q} \leq 1\}$,
where $a(s) = (z_1(s), \pm\sqrt{1-|z_1(s)|^2}, \frac{1}{\sqrt{1-|z_1(s)|^2}} z_2(s))$ with

$$\begin{aligned} z_1(s) &= -\cos^2\theta e^{i(2ps)} - \sin^2\theta e^{i(2qs)}, \\ z_2(s) &= \sin\theta\cos\theta(e^{i(2ps)} - e^{i(2qs)}), \\ z_3(s) &= -\sin^2\theta e^{i(2ps)} - \cos^2\theta e^{i(2qs)}, \end{aligned}$$

and $\theta \in (0,\pi) - \{\frac{\pi}{2}\}$ satisfying $C_i = P_\theta E_i P_\theta^t (i = 1,2)$.

Theorem 4.3. [9] *Let* $A \in \mathrm{Sym}(2,\mathbb{R})$ *be a* 2×2 *symmetric matrix which has the spectral decomposition* $A = (rp)C_1 + (rq)C_2$, *where* $r > 0$ *is a real number and* p, q *are coprime integers. Let* $\theta \in [0,\pi]$ *such that* $A = P_\theta A_0 P_\theta^t$.

(1) *If both p and q are odd integers, or equivalently, the geodesic* $\alpha(t,A)$ *in* $\mathcal{M}$ *is symmetric, then* $\hat{\kappa}(A_0)$ *is a link of* 2*-components in* $S^2 \times S^1$ *which has period* 2 *of type* $(S^1 \times S^1, T(|p|, \frac{|p+q|}{2}) \,\dot{\cup}\, T(|p|, \frac{|p+q|}{2}))$.
(2) *If one of p and q is an even integer, or equivalently, the geodesic* $\alpha(t,A)$ *in* $\mathcal{M}$ *is not symmetric, then* $\hat{\kappa}(A_0)$ *is a knot in* $S^2 \times S^1$ *which has period* 2 *of type* $(S^1 \times S^1, T(|p|, |p+q|))$.

(3) *If* $\theta \neq 0, \frac{\pi}{2}, \pi$, *then* $\hat{\kappa}(A)$ *is a link in* $S^2 \times S^1$ *which has period* 2 *of type* $(S^1 \times S^1, 2|p-q|$ *points*$)$.

(4) $\hat{\kappa}(A_0)$ *is a link in* $S^2 \times S^1$ *which has period* 2 *of type* $(S^1 \dot{\cup} S^1, \emptyset)$.

Problem 4.4. *Realize the closed geodesics on the real conformal compactification of the space* $\mathrm{Sym}(2, \mathbb{R})$ *as classical knots or links in* S^3 *and classify these knots or links as certain types of symmetries of periodic knots.*

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