Trends in Mathematics
Information Center for Mathematical Sciences
Volume 3, December 2000, Pages 35–42

NONTRIVIAL SOLUTIONS OF A NONLINEAR BIHARMONIC EQUATION

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Abstract. Let Ω be a smooth bounded region in R^n with smooth boundary $\partial\Omega$. We study the multiplicity of solutions of the nonlinear biharmonic equation with Dirichlet boundary condition, $\Delta^2 u + c\Delta u = g(u)$ in Ω, where $c \in R$ and Δ^2 denote the biharmonic operator. We show that the equation has nontrivial solutions when $g : R \to R$ is a differentiable function tith some condition such that $g(0) = 0$.

1. Introduction

In this paper we study the multiplicity of solutions of the nonlinear biharmonic equation with Dirichlet boundary condition

$$\begin{aligned} \Delta^2 u + c\Delta u &= g(u) \qquad \text{in } \Omega, \\ u = 0, \qquad \Delta u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a smooth bounded region in R^n with smooth boundary $\partial\Omega$ and Δ^2 denote the biharmonic operator. Here we assume that $g : R \to R$ is a differentiable function such that $g(0) = 0$ and

$$g'(\infty) = \lim_{|u|\to\infty} \frac{g(u)}{u} \in R.$$

Let λ_k, $k \geq 1$ denote the eigenvalues and ϕ_k, $k \geq 1$ the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem

$$\begin{aligned} \Delta u + \lambda u &= 0 \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where each eigenvalue λ_k is repeated as often as its multiplicity. We recall that $0 < \lambda_1 \leq \lambda_2 \leq \ldots \to +\infty$, and that $\phi_1(x) > 0$ for $x \in \Omega$.

In [8] the authors studied the fourth order nonlinear elliptic equation with jumping nonlinearity

$$\begin{aligned} \Delta^2 u + c\Delta u &= bu^+ + s \qquad \text{in } \Omega, \\ u = 0, \qquad \Delta u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where $u^+ = \max\{u, 0\}$. This kind of jumping nonlinearity furnishes a good moddel to study travelling waves in a suspension bridge [9]. In [4] the authors also studied the nonlinear wave equation with this kind of jumping nonlinearity. Nonlinear

2000 *Mathematics Subject Classification.* 35J40.
Key words and phrases. Multiple solutions, biharmonic equation, variational reduction method, degree theory.
Received June 30, 2000.

biharmonic equation (1.2) is also interesting when the jumping nonlinearity $bu^+ + s$ is replaced by a somewhat more general function $g(u)$.

Now we state the main result of this paper.

THEOREM 1.1. *Let $c < \lambda_1$. If $g'(0) < \lambda_1(\lambda_1 - c)$, $g'(\infty) \in (\lambda_k(\lambda_k - c), \lambda_{k+1}(\lambda_{k+1} - c))$ with $k \geq 2$, and $0 < g'(t) \leq \gamma < \lambda_{k+1}(\lambda_{k+1} - c)$. then (1.1) has at least three solutions.*

2. A Variational Reduction Method

We assume that $c < \lambda_1$. Let us denote an element u in $L^2(\Omega)$ as

$$u = \sum h_k \phi_k, \qquad \sum h_k^2 < \infty.$$

Now, we define a subspace H of $L^2(\Omega)$ as follows

$$H = \{u \in L^2(\Omega) : \sum |\lambda_k(\lambda_k - c)| h_k^2 < \infty\}.$$

Then this is a complete normed space with a norm

$$|||u||| = [\sum |\lambda_k(\lambda_k - c)| h_k^2]^{\frac{1}{2}}.$$

Since $\lambda_k \to +\infty$ and c is fixed, we have (see [8]):
(i) $\Delta^2 u + c\Delta u \in H$ implies $u \in H$.
(ii) $|||u||| \geq C\|u\|_{L^2(\Omega)}$, for some $C > 0$.
(iii) $\|u\|_{L^2(\Omega)} = 0$ if and only if $|||u||| = 0$.

We assume that g is differentiable, $g'(0) < \lambda_1(\lambda_1 - c)$, $g'(\infty) \in (\lambda_k(\lambda_k - c), \lambda_{k+1}(\lambda_{k+1} - c))$, and $0 < g'(t) \leq \gamma < \lambda_{k+1}(\lambda_{k+1} - c)$. From the assumptions of g there exists $a > 0$ such that $|g(u)| \leq a(1 + |u|)$.

LEMMA 2.1. *All solutions in $L^2(\Omega)$ of*

$$\Delta^2 u + c\Delta u = g(u) \qquad \text{in } L^2(\Omega)$$

belong to H.

Proof. Since $|g(u)| \leq a(1 + |u|)$ for some $a > 0$, we have $g(u) \in L^2(\Omega)$. Let $g(u) = \sum h_k \phi_k \in L^2(\Omega)$. Then

$$(\Delta^2 + c\Delta)^{-1}(g(u)) = \sum \frac{1}{\lambda_k(\lambda_k - c)} h_k \phi_k.$$

Hence we have

$$|||(\Delta^2 + c\Delta)^{-1} g(u)|||^2 = \sum |\lambda_k(\lambda_k - c)| \frac{1}{(\lambda_k(\lambda_k - c))^2} h_k^2 \leq C \sum h_k^2$$

for some $C > 0$, which means that

$$|||(\Delta^2 + c\Delta)^{-1} g(u)||| \leq C\|u\|_{L^2(\Omega)}.$$

∎

With the aid of Lemma 2.1 it is enough that we investigate the existence of solutions of (1.1) in the subspace H of $L^2(\Omega)$. Let us define the functional in $H \times R$,

$$I(u) = \int_\Omega \frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - G(u),$$

where $G(u) = \int_0^s g(\sigma)d\sigma$. Then $I(u)$ is well defined. The solutions of (1.1) coincide with the critical points of $I(u)$.

PROPOSITION 2.1. *Assume that $g(u)$ satisfies the conditions of Theorem 1.1. Then $I(u)$ is continuous and Frechét differentiable in H and*

$$DI(u)(h) = \int_\Omega \Delta u \cdot \Delta h - c\nabla u \cdot \nabla h - g(u)h \tag{2.1}$$

for $h \in H$.

For the sake of completeness we recall that if I is a function of class C^1 and u_0 is a critical point of I, then u_0 is called of mountain pass type if for every open neighborhood U of u_0, $I^{-1}(-\infty, I(u_0)) \cap U \neq \emptyset$ and $I^{-1}(-\infty, I(u_0)) \cap U \backslash \{u_0\}$ is not pass connected.

Let V be k dimensional subspace of H spanned by $\phi_1, \ldots, \phi_k$ whose eigenvalues are $\lambda_1(\lambda_1 - c), \ldots, \lambda_k(\lambda_k - c)$. Let W be the orthogonal complement of V in H. Let $P : H \to V$ be the orthogonal projection of H onto V and $I - P : H \to W$ denote that of H onto W. Then every element $u \in L^2(\Omega)$ is expressed by $u = v + z$, $v \in Pu$, $z = (I - P)u$. Then (1.1) is equivalent to the system with two unknowns v and z:

$$\Delta^2 v + c\Delta v = P(g(v+z)),$$
$$\Delta^2 z + c\Delta z = (I-P)(g(v+z)).$$

LEMMA 2.2. *Let $c < \lambda_1$. Assume that $g'(0) < \lambda_1(\lambda_1 - c)$, $g'(\infty) \in (\lambda_k(\lambda_k - c), \lambda_{k+1}(\lambda_{k+1} - c))$, and $0 < g'(t) \leq \gamma < \lambda_{k+1}(\lambda_{k+1} - c)$, $k \geq 2$. Then we have:*
(i) For any fixed $v \in V$ there are $m > 0$ and $\alpha > 1$ such that

$$(DI(v+w) - DI(v+w_1), w - w_1) \geq m\|w - w_1\|^\alpha, \qquad \text{for all } w \in W,\ w_1 \in W.$$

(ii) There exists a unique solution $z \in W$ of the equation

$$\Delta^2 z + c\Delta z = (I-P)(g(v+z)) \qquad \text{in } W. \tag{2.3}$$

If we put $z = \theta(v)$, then θ is continuous on V and satisfies a uniform Lipschitz condition in v with respect to the L^2 norm(also norm $\|| \cdot \||$). Moreover

$$DI(v + \theta(v))(w) = 0 \qquad \text{for all } w \in W,$$

and

$$I(v + \theta(v)) = \min_{w \in W} I(v+w). \tag{2.4}$$

(iii) If $\tilde{I} : V \to R$ is defined by $\tilde{I}(v) = I(v + \theta(v))$, then $\tilde{I}$ has a continuous Fréchet derivative $D\tilde{I}$ with respect to v, and

$$D\tilde{I}(v)(h) = DI(v + \theta(v))(h) \qquad \text{for all } h \in V.$$

(iv) If $v_0 \in V$ is a critical point of $\tilde{I}$ if and only if $v_0 + \theta(v_0)$ is a critical point of I.
(v) Let $S \subset V$ and $\Sigma \subset H$ be open bounded regions such that

$$\{v + \theta(v); v \in S\} = \Sigma \cap \{v + \theta(v); v \in V\}.$$

If $D\tilde{I}(v) \neq 0$ for $v \in \partial S$, then

$$d(D\tilde{I}, S, 0) = d(DI, \Sigma, 0),$$

where d denote the Leray-Schauder degree.
(vi) If $u_0 = v_0 + \theta(v_0)$ is a critical point of mountain pass type of I, then v_0 is a critical point of mountain pass type of $\tilde{I}$.

3. A Degree Theory Applied to P.D.E

Let $\gamma > \lambda_1(\lambda_1 - c)$ and $q(\gamma) = q$ be the homogeneous function defined by

$$q(x) = \begin{cases} \gamma x & \text{for } x \geq 0, \\ g'(0)x & \text{for } x < 0. \end{cases}$$

Let Q be the primitive of q with $Q(0) = 0$, and $J : H \to R$ be the functional defined by

$$J(u) = \int_\Omega [\frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - Q(u)]dx. \tag{3.1}$$

Then $J(u)$ is well-defined, Fréchet differentiable in H with

$$DJ(u)h = \int_\Omega (\Delta^2 u + c\Delta u - q(u))h dx,$$

and its critical points are the weak solutions of the problem

$$\Delta^2 u + c\Delta u = q(u) \qquad \text{in } \Omega, \tag{3.2}$$

$$u = 0, \Delta u = 0 \qquad \text{on } \partial\Omega.$$

Since $0 < g'(0) < \lambda_1(\lambda_1 - c)$ and the principal eigenvalue of $\Delta^2 + c\Delta$ in any subregion of Ω is bigger than or equal to $\lambda_1(\lambda_1 - c)$, we see that if $u \neq 0$ is a weak solution of (3.2) then u is a positive eigenfunction. Since this contradicts that $\gamma > \lambda_1(\lambda_1 - c)$, we conclude that $u = 0$ is the only critical point of J.

Lemma 3.1. *If B is a ball in H containing zero, then*

$$d(DJ, B, 0) = 0.$$

Proof. Let Z be the subspace spanneded by $\phi_1, \dots, \phi_k$ with k big enough. Then by the definition of the Leray-Schauder degree

$$d(DJ, B, 0) = d(PDJ, B \cap Z, 0),$$

where P denotes the orthogonal projection onto Z. Since $\gamma > \lambda_1(\lambda_1 - c)$, $h(t) = q(t) - \lambda_1(\lambda_1 - c)t > 0$ for $t \neq 0$. Since $\phi_1 > 0$ in Ω and is in Z, we have

$$\begin{aligned} (PDJ(u), \phi_1) = (DJ(u), \phi_1) &= \int_\Omega [\Delta^2 u \cdot \phi_1 + c\Delta u \cdot \phi_1 - \lambda_1(\lambda_1 - c)u - h(u)\phi_1]dx \\ &= \int_\Omega -h(u)\phi_1 dx < 0 \qquad \text{if } u \in Z \cap \partial B. \end{aligned}$$

Thus for each $s \in [0,1]$ and $u \in Z \cap \partial B$,

$$(sPDJ(u) + (1-s)(-\phi_1), \phi_1) < 0.$$

Hence by homotopy invariance of the Leray-Schauder degree we have

$$d(PDJ, B \cap Z, 0) = d(-\phi_1, B \cap Z, 0) = 0.$$

So the lemma is proved. ∎

Let g^+ be the function defined by

$$g^+(x) = \begin{cases} g(x) & \text{if } x \geq 0, \\ g'(0)x & \text{if } x < 0. \end{cases}$$

Let $G^+(x) = \int_0^x g^+(\sigma)d\sigma$, and $I^+ : H \to R$ be the functional defined by

$$I^+(u) = \int_\Omega [\frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - G^+(u)]dx. \tag{3.2}$$

Then I^+ satisfies the hypothesis of the Mountain Pass Theorem:

LEMMA 3.2. *$I^+(u)$ satisfies the hypothesis of the Mountain Pass Theorem.*

Proof. The hypothesis of mountain pass theorem is the following:
(i) $I^+ \in C^1(H, R)$ and $I^+(0) = 0$,
(ii) there are constants $\rho, \alpha > 0$ such that $I^+ |_{\partial B_\rho} \geq \alpha$, and
(iii) there is an $e \in E \setminus \bar{B}_\rho$ such $I^+(e) \leq 0$.
(iv) I^+ satisfies (PS) condition.

By the same method of the proof of Proposition 2.1 we can prove that $I^+ \in C^1(H, R)$. We note that $I^+(0) = 0$. So (i) is satisfied.
Since $g(0) = 0$, g is differentiable, and $0 < g'(t) \leq \gamma$, given any $\epsilon > 0$ there is a $\delta > 0$ such that $|t| \leq \delta$ implies

$$G^+(t) \leq \frac{1}{2}\epsilon|t|^2$$

for all $t \in R$. Thus

$$|\int_\Omega G^+(u)dx| \leq \frac{\epsilon}{2}\int_\Omega u^2 dx = \frac{\epsilon}{2}\|u\|^2_{L^2(\Omega)} \leq a_1\frac{\epsilon}{2}|||u|||^2$$

for some $a_1 > 0$. Since ϵ is arbitrary, $\int_\Omega G^+(u)dx = o(|||u|||^2)$ as $u \to 0$. Thus

$$\begin{aligned} I^+(u) &= \int_\Omega (\frac{1}{2}(\Delta^2 u + c\Delta u) - G^+(u)) \\ &= \frac{1}{2}|||u|||^2 - \int_\Omega G^+(u) = \frac{1}{2}|||u||| + o(|||u|||^2). \end{aligned}$$

as $u \to 0$. So (ii) is satisfied.
By the same method of the proof of [4] (iii), (iv) can be proved. ∎

By Lemma 3.2, I^+ has a critical point u^+ of mountain pass type. Therefore if the set of critical points of I^+ is discrete, then at least one of them is of mountain pass type. We can state the above discussion as follows:

LEMMA 3.3. *I^+ has at least one critical point u^+ of mountain pass type.*

LEMMA 3.4. *Let B_r be the sphere centered at 0 with radius $r > 0$. Then*

$$d(DI^+, B_r, 0) = 0$$

for r large enough.

Proof. Let $\gamma = g'(\infty)$ and J be the functional in (3.1). Since $g'(\infty)$ is not an eigenvalue of $\Delta^2 + c\Delta$ with Dirichlet boundary conditions, for $r > 0$ large enough and $s \in [0,1]$ the function

$$sDI^+ + (1-s)DJ$$

has no zero on ∂B_r. Thus by invariance under homotopy of the Leray-Schauder degree

$$d(DI^+, B_r, 0) = d(DJ, B_r, 0) = 0.$$

∎

LEMMA 3.5. *0 is an isolated local minimum point of I^+ and I. Thus*

$$d(DI^+, B, 0) = 1 = d(DI, B, 0),$$

where B is a ball centered at zero containing no other critical point.

Proof. Since g is differentiable and $g(0) = 0$, there exists $0 < t < s$ such that $g(s) = sg'(t)$. Thus for $\xi > 0$, $G^+(\xi) = \int_0^\xi g(s)ds = \frac{1}{2}g'(t)\xi^2$. Thus for $u > 0$ we have

$$\begin{aligned} I^+(u) &= \int_\Omega [\frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - G^+(u)]dx \\ &= \frac{1}{2}|||u|||^2 - \int_\Omega \frac{1}{2}g'(t)u^2 dx \geq \frac{1}{2}\{\lambda_k(\lambda_k - c) - g'(t)\}\|u\|^2_{L^2(\Omega)} > 0, \end{aligned}$$

since $|||u|||^2 \geq \lambda_k(\lambda_k - c)\|u\|^2_{L^2(\Omega)}$ and $0 < g'(t) \leq \lambda_k(\lambda_k - c)$. For $u < 0$ we have

$$I^+(u) = \frac{1}{2}|||u|||^2 - \frac{1}{2}\int_\Omega g'(0)u^2 dx \geq \frac{1}{2}\{\lambda_k(\lambda_k - c) - g'(0)\|u\|^2_{L^2(\Omega)}\} > 0,$$

Since $g'(0) < \lambda_1(\lambda_1 - c) < \lambda_k(\lambda_k - c)$. From $I^+(0) = 0$ it follows that 0 is a local minimum point of I. ∎

Next we will prove that there exists an local maximum point $u_0 \in H$ such that $DI(u_0) = 0$ and, if isolated, the local Leray-Schauder degree is $(-1)^k$.

LEMMA 3.6. *Under the assumptions of Theorem 1.1, I has a local maximum critical point u_0 such that, if isolated,*

$$d(DI, C, 0) = (-1)^k,$$

where C is any region containing no other critical point of I.

Proof. By Lemma 2.2 ther exists $\theta : V \to W$ such that $I(v+\theta(v)) = \min_{w \in W} I(v+w)$. Moreover, $\theta(v)$ is the unique member of W such that $(DI(v + \theta(v)), w) = 0$ for all $w \in W$, and the functional $\tilde{I} : V \to R$ defined by $\tilde{I}(v) = I(v + \theta(v))$ is of class C^1 and $(D\tilde{I}(v), h) = (DI(v + \theta(v)), h)$ for all $v, h \in V$. Since $g'(\infty) \in (\lambda_k(\lambda_k - c), \lambda_{k+1}(\lambda_{k+1} - c))$ there exists $a \in R$ and $\bar{\gamma} > \lambda_k(\lambda_k - c)$ such that $G(s) \geq \bar{\gamma}\frac{s^2}{2} + a$. Hence for $v \in V$

$$I(v) = \int_\Omega (\frac{1}{2}|\Delta v|^2 - \frac{c}{2}|\nabla v|^2 - G(v)) \leq \frac{1}{2}|||v|||^2 - \frac{\bar{\gamma}}{2}\int_\Omega v^2 - a|\Omega|.$$

Since $|||v|||^2 \leq \lambda_k(\lambda_k - c)\|v\|^2_{L^2(\Omega)}$ for $v \in V$, we have

$$I(v) \leq \frac{1}{2}\{1 - \frac{\bar{\gamma}}{\lambda_k(\lambda_k - c)}\}|||v|||^2 - a|\Omega| \longrightarrow -\infty \qquad \text{as } |||v||| \to \infty.$$

Since $\tilde{I}(v) = I(v+\theta(v)) = \min_{w\in W} I(v+w) \le I(v)$, $\tilde{I}(v) \to -\infty$ as $|||v||| \to \infty$. Since $\dim V < \infty$ ther exsts $v_0 \in V$ such that

$$\tilde{I}(v_0) = \max_{v\in V} \tilde{I}(v)$$

and $D\tilde{I}(v_0) = 0$. If we take $u_0 = v_0 + \theta(v_0)$, then u_0 is a critical point of $I(u)$. If v_0 is isolated, then u_0 is also isolated. Let us take $Z = \{v \in V; v+\theta(v) \in C\}$. Since $-\tilde{I}$ has a local minimum at v_0, $d(D\tilde{I}, Z, 0) = (-1)^k$. ∎

4. Proof of the Main Result

Suppose k is even. Let R be large enough so that if $D\tilde{I}(v) = 0$, then $|||v||| < R$. Since $0 < g'(t) \le \gamma < \lambda_{k+1}$, there exist a_1 and a_2 such that for all $v \in V$ $|||\theta(v)||| \le a_1 + a_2|||v|||$. Thus if $u = v + w$ is a critical point of I, then $|||v||| \le R$ and $|||w||| \le C_1 + C_2|||v|||$. Since $-\tilde{I}$ is coercive, $d(D\tilde{I}, B_R, 0) = (-1)^k = 1$. Thus by Lemma 2.2 $d(DI, U, 0) = 1$, where $U = \{v + w : |||v||| \le R, |||w||| \le C_1 + C_2R\}$. Suppose that K, the set of critical points of J is finite. Let U_1 and U_2 be the disjoint open bounded regions in H such that $\bar{U_1} \cap K = \{0\}$, $\bar{U_2} \cap K = \{u_0\}$. By the excision property of the Leray-Schauder degree we have

$$\begin{aligned} 1 = d(DI, U, 0) &= d(DI, U_1, 0) + d(DI, U_2, 0) + d(DI, U - \overline{(U_1 \cup U_2)}, 0) \\ &= 1 + 1 + d(DI, U - \overline{(U_1 \cup U_2)}, 0) \end{aligned}$$

Thus, by the existence property of the Leray-Schauder degree there exists $u_1 \in U - \overline{(U_1 \cup U_2)}$ such that $DI(u_1) = 0$ and $d(DI, U - \overline{(U_1 \cup U_2)}, 0) = -1$. We know that $U - \overline{(U_1 \cup U_2)}$ contains a critical point of mountain pass type. If u_1 is a critical point of mountain pass type, (1.1) has at least three solutions. If u_1 is not a critical point of mountain pass type, there exists a critical point of mountain pass type $u_2 \in U - \overline{(U_1 \cup U_2)}$, which proves that (1.1) has at least three solutions.
Suppose k is odd. Let U and U_i, $i = 1, 2$ be as above. Then we have $d(DI, U, 0) = -1$, $d(DI, U_1, 0) = 1$ and $d(DI, U_2, 0) = -1$. Thus by the excision property of the Leray-Schauder degree

$$-1 = d(DI, U, 0) = 1 - 1 + d(DI, U - \overline{(U_1 \cup U_2)}, 0).$$

Thus, by the existence property of the Leray-Schauder degree there exists $u_1 \in U - \overline{(U_1 \cup U_2)}$ such that $DI(u_1) = 0$ and $d(DI, U - \overline{(U_1 \cup U_2)}, 0) = -1$. Similar method as the case k is even (1.1) has at least three solutions. ∎

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