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ON THE EXISTENCE AND MULTIPLICITY OF POSITIVE RADIAL SOLUTIONS FOR SEMILINEAR ELLIPTIC PROBLEMS

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Abstract. We introduce recent results on semilinear elliptic problems of Gelfand or Emden-Fowler type. The survey focus on the problems on annuli or exterior domains.

1. Introduction

The aim of this paper is to survey the recent development of existence and multiplicity results for positive radial solutions of semilinear elliptic problems of the form;

$$\text{(P)} \qquad \begin{aligned} &\triangle u + \lambda k(|x|)f(u) = 0, \text{ in } \Omega \\ &u|_{\partial\Omega} = 0, \end{aligned}$$

where λ is a positive real parameter. Problems of this type arise in a variety of interesting applications. For example, the generalized Gelfand equation with variant forms of $f(u) = e^u$ appear as equilibrium solutions for models of gas diffusion through porous media [ACP], in the context of the theory of thermal self-ignition of a chemically active mixture of gasses in a vessel [BK], [CL], [G], [P]. Specially, the case $f(u) = \alpha(1+\beta-u)e^{\frac{\gamma}{u}}$ for γ large, has been studied in [D] in connection with models for catalysis, $f(u) = e^{\frac{-1}{|u|}}$ was studied in [P] for its connection with models for chemically reacting systems, and $f(u) = b(c-u)e^{\frac{-K}{1+u}}$ was treated in [C] as a model for an adiabatic tubular reactor. The generalized Emden-Fowler or Thomas-Fermi equations with variant forms of $f(u) = u^p$, $p > 0$ arise in the fields of gas dynamics, nuclear physics and chemically reacting systems [W] or in the study of atomic structures [LP] and atomic calculations [Cs]. In this article, we restrict our survey for which Ω is an annulus or an exterior domain.

2. On an Annulus

Let $\Omega = \{x \in \mathbf{R}^n | R_1 < |x| < R_2\}$ for $R_1, R_2 > 0$. Looking for radial solutions, we apply consecutive change of variables, $r = |x|$, $s = \int_r^{R_2} t^{1-n}dt$ and $t = \frac{m-s}{m}$, where $m = -\int_{R_1}^{R_2} t^{1-n}dt$, to transform the problem into second order o.d.e. of the form;

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$$(s_a) \qquad \begin{aligned} &z''(t) + \lambda h(t) f(z(t)) = 0 \\ &z(0) = 0 = z(1), \end{aligned}$$

where the coefficient function h is given by

$$h(t) = m^2 (R_2^{-(n-2)} - (n-2)m(1-t))^{\frac{-2(n-1)}{n-2}} k(R_2^{-(n-2)} - (n-2)m(1-t)).$$

We notice that $h : [0,1] \to [0,\infty)$ is continuous so that problem (S_a) is regular. The first interesting results was proved in 1987 by Garaizar [Ga] and Bandle, Coffman and Marcus [BCM] for the case $k(r) \equiv 1$.

Theorem 1. (Garaizar) *Assume*
(A_1) $f \in C[0,\infty)$, $f(0) = 0$, $f(u) > 0$ *for* $u > 0$.
(A_2) $f_o \triangleq \lim_{u\to\infty} \frac{f(u)}{u} = 0$.
$(A_3)'$ *there exists* $b, d_1, d_2 > 0$ *and* $k > 1$ *such that* $d_1 u^k \le f(u) \le d_2 u^k$ *for* $u \ge b$.
Then problem (P) has at least one positve radial solution for all $\lambda > 0$ *and for all* R_1 *and* R_2 *such that* $0 < R_1 < R_2 < \infty$.

Theorem 2. (Bandle, Coffman and Marcus)
Let $f \in C^1[0,\infty)$ *and satisfy* (A_1). *Assume* (A_2) *and also*
(A_3) $f_\infty \triangleq \lim_{u\to\infty} \frac{f(u)}{u} = \infty$.
(A_4) f *is nondecreasing on* $(0,\infty)$.
Then problem (P) has at least one positve radial solution for all $\lambda > 0$ *and for all* R_1 *and* R_2 *such that* $0 < R_1 < R_2 < \infty$.

In 1989, Coffman and Marcus [CM], Bandle and Kwang [BaK] and Lin [L] proved that the hypothesis (A_4) in Theorem 2 is not necessary for existence. For example, Lin proved the followings.

Theorem 3. *Assume* (A_2) *and* (A_3). *Assume also*
$(A)'$ $k \in C^1(0,\infty)$, $k(r) \ge 0$ *for* $r > 0$ *and is not identically zero in any finite subinterval of* $(0,\infty)$.
Then problem (P) has at least one positve radial solution for all $\lambda > 0$ *and for all* R_1 *and* R_2 *such that* $0 < R_1 < R_2 < \infty$.

There arise a quastion that f is sublinear (i.e. $f_o = \infty$ and $f_\infty = 0$) implies problem (P) has a positve radial solution for any annulus. Obviously the previous results like Theorem 1 cannot deal with it. For example, let

$$f(u) = \begin{cases} u^{\frac{1}{2}}, & u \le 1 \\ (-1 + \ln 2)(u-1) + 1, & 1 < u < 2 \\ \ln u, & u \ge 2 \end{cases}$$

Then $f_o = \infty$ and $f_\infty = 0$. However, there does not exist $k > 1$ such that $(A_2)'$ in Theorem 1 holds for this function. Wang [Wa] gave a positive answer to this question.

Theorem 4. *Assume*
(A) $k \in C[0,\infty), k(r) \geq 0$ *for* $r \geq 0$ *and is not identically zero in finite subinterval of* $(0,\infty)$.
(A)' $f \in C[0,\infty),\ f(u) \geq 0$ *for* $r \geq 0$.
(S_2) $f_o = \infty$.
(S_3) $f_\infty = 0$.
Then problem (P) has at least one positve radial solution for all $\lambda > 0$ *and for all* R_1 *and* R_2 *such that* $0 < R_1 < R_2 < \infty$.

Erbe, Hu and Wang [EHW] considered the case of the superlinearity of f at one end and sublinearity of f at the other end and obtained the multiplicity of positive radial solutions.

Theorem 5. *Assume* (S_2) *and* (A_3). *Assume that there is* $p > 0$ *such that for all* $r \in [R_1, R_2]$ *and* $0 \leq u \leq p$

$$k(r)f(u) \leq b(\frac{R_2^n - R_1^n}{2-n})^2 p$$

Then the problem has at least two positive radial solutions.

Questions for the cases that k is not continuous or f is neither super linear nor sublinear were first answered by Lan and Webb [LW] in 1998. Among several theorems in [LW], they proved;

Theorem 6. *Assume*
(A)'' $k : [0,\infty) \to [0,\infty]$ *satisfies that for any* $R_1 < R_2$, *there exist at most* $r^1, \ldots, r^m \in [R_1, R_2]$ *such that* $k : [R_1, R_2] \setminus \{r^1, \ldots, r^m\} \to (0,\infty)$ *is continuous,* $\int_{R_1}^{R_2} k(r)dr$ *exists and* k *is not identically zero in* $[R_1, R_2]$. *Also assume either* (A_2) *and* (A_3) *or* (S_2) *and* (S_3).
Then problem (P) has at least one positve radial solution for all $\lambda > 0$ *and for all* R_1 *and* R_2 *such that* $0 < R_1 < R_2 < \infty$.

3. On an Exterior Domain

Let $\Omega = \{x \in \mathbf{R}^n : |x| > r_o\}$, then Dirichlet boundary problem (P) can be written as follows;

(P) $$\triangle u + \lambda k(|x|)f(u) = 0, \text{ in } \Omega$$

(D) $$u(x) = 0, \text{ if } |x| = r_o$$
$$u \to 0, \text{ as } |x| \to \infty$$

Through change of variables, $r = |x|$ and $t = 1 - (\frac{r}{r_o})^{2-n}$, we may transform (P) into second order o.d.e. of the form;

(S_e) $$u''(t) + \lambda q(t) f(u(t)) = 0$$
$$u(0) = 0 = u(1),$$

where the coefficient function q can be given by

$$q(t) = \frac{r_o^2}{(n-2)^2}(1-t)^{\frac{-2(n-1)}{n-2}} k(r_o(1-t)^{\frac{-1}{n-2}}).$$

We notice that $q \in C[0,1)$ is singular at $t = 1$ and also notice that the coefficient function of problems on an annulus is regular on [0,1] and this is the main difference between problems on annuli and exterior domains. The difficulty for singular problems comes as follows; The fixed point operator T of problem (S_e) is given by

$$Tu(t) = \lambda \int_0^1 G(t,s)q(s)f(u(s))ds$$

which maps the cone of nonnegative continuous functions on [0,1] into itself. Thus it is continuous by Lebesgue dominated convergence theorem and

$$(Tu)'(t) = -\lambda \int_0^t sq(s)f(u(s)ds + \lambda \int_t^1 (1-s)q(s)f(u(s))ds.$$

If it happens that q is continuous on $[0,1]$ like the problems on annular domains, then clearly T is a compact mapping. But if $q \notin L^1[0,1)$, then the second integral may grow without bound as t approaches 1 from the right which hardly make T compact. Nevertheless, we claim that with the hypothesis $(1-t)q(t) \in L^1[0,1)$, the operater T is compact. We notice that the corresponding hypothesis on k for condition $(1-t)q(t) \in L^1[0,1)$ is $\int_{r_o}^\infty rk(r)dr < \infty$.
The survey in this section consists of two folds, one of Gelfand type and the other of Emden-Fowler type. The former type is problems with the nonlinearity $f(u) = e^u$ or its variants and the latter is $f(u) = u^p$ or its variant. Throughout this section, we assume, for convenience, $k \in C([r_o,\infty),(0,\infty))$ and $f \in C([0,\infty),[0,\infty))$. Results for Gelfand type were recently initiated by Choi [Ch] in 1991 precisely for the case $f(u) = e^u$.

Theorem 7. *Assume* $k \in C^1[0,\infty)$.
(H) $\int_{r_o}^\infty rk(r)dr < \infty$.
Then there exists $\lambda^* > 0$ *such that problem (P), (D) has at least one positive radial solution for* $\lambda \in (0,\lambda^*)$ *and none for* $\lambda \in (\lambda^*,\infty)$.

The nonlinear term $f(u) = e^u$ was generalized to several directions by Wong [Wo], Fink- Gatica-Hermandez [FGH], Dalmasso [Da] and Ha-Lee [HL]. For example, Dalmasso proved the followings;

Theorem 8. *Assume* (H)
(H_1) f *is nonincreasing and* $f(0) > 0$.
(H_2') *There exists* $d > 0$ *such that* $f(z) \geq dz, \quad \forall z \geq 0$.
Then there exists $\lambda^* > 0$ *such that problem (P), (D) has at least one positive radial solution for* $\lambda \in (0,\lambda^*)$ *and none for* $\lambda \in (\lambda^*,\infty)$.

Condition (H_1) could be released in Ha-Lee [HL] as follows;

Theorem 9. *Assume (H)*
(H_2'') $f(u) \geq e^u, \quad \forall u \geq 0$.
Then there exists $\lambda^* > 0$ *such that problem (P), (D) has at least one positive radial solution for* $\lambda \in (0,\lambda^*)$ *and none for* $\lambda \in (\lambda^*,\infty)$.

In [C], Choi gave a conjecture, with a numerical diagram, on the existence of the second solution for $\lambda \in (0,\lambda^*)$ and it was first proved by Ha-Lee [HL] and generalized by Lee [Lee2].

Theorem 10. *Assume* (H) *and* (H_1).
(H_2) $\lim_{u\to\infty} \frac{f(u)}{u} = 0$.
Then there exists $\lambda^* > 0$ *such that problem (P), (D) has at least two positive radial solutions, at least one positive solution or none according to* $\lambda \in (0, \lambda^*)$, $\lambda = \lambda^*$ *or* $\lambda \in (\lambda^*, \infty)$.

Recent developement of Emden-Fowler type problems was initiated by Zhang [Z] in 1994. He concerned with the sublinear case, precisely, $f(u) = u^p$, $0 < p < 1$. Here we give a generalized version of Zhang's result. We assume $f(0) = 0$ for Thm 11 through Thm 13.

Theorem 11. *Assume*
(S_1) $\lim_{u\to 0} \frac{f(u)}{u} = \infty$.
(S_2) $\lim_{u\to\infty} \frac{f(u)}{u} = 0$.
Then the problem (P), (D) has a positive radial solution for all $\lambda > 0$ *and all* $r_o > 0$.

Superlinear case was proved by Lee [Lee1].

Theorem 12. *Assume* (H) *and* (H_2).
(H_1') $\lim_{u\to 0^+} \frac{f(u)}{u} = 0$.
Then the problem (P), (D) has a positive radial solution for all $\lambda > 0$ *and all* $r_o > 0$.

In connection with the study of problem (P) in whole domain R^n, it is interesting to consider the problem with the following pertubed boundary condition;

$$\text{(T)} \qquad \begin{aligned} &u(x) = \alpha > 0, \quad \text{if } |x| = r_o \\ &u(x) \to 0, \quad \text{as } |x| \to \infty \end{aligned}$$

It is interesting to notice that the result (Thm 13) of problem (P), (T) is quite differnt from that (Thm 12) of problem (P), (D), they have quite similar nonlinearity though.

Theorem 13. ([Lee3]) *Assume* $(H), (H_2)$.
(H_1'') f *is strictly increasing.*
Then there exist $0 < \lambda_o \le \lambda^*$ *such that problem (P), (T) has at least two positive radial solutions, at least one positive radial solution or none according to* $\lambda \in (0, \lambda_o)$, $\lambda \in [\lambda_o, \lambda^*]$ *or* $\lambda \in (\lambda^*, \infty)$.

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