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EFFECT OF SYMMETRY TO THE STRUCTURE OF POSITIVE SOLUTIONS IN NONLINEAR ELLIPTIC PROBLEMS

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Abstract. We consider the problem :

$$\begin{aligned} \Delta u + u^p = 0 &\quad \text{in} \quad \Omega_R \\ u = 0 &\quad \text{on} \quad \partial\Omega_R \\ u > 0 &\quad \text{in} \quad \Omega_R, \end{aligned}$$

This problem is invariant under the orthogonal coordinate transformations, in other words, $O(N)$-symmetric. It will be demonstrated how the symmetry affects to the structure of positive solutions. In fact, for a clased subgroup G of $O(N)$, we consider a natural group action $G \times S^{N-1} \to S^{N-1}$, and we show that a structure of the orbits space affects to the sturcture of the positive solutions.

We consider the following problem

$$\begin{aligned} \Delta u + u^p = 0 &\quad \text{in} \quad \Omega_R \\ u = 0 &\quad \text{on} \quad \partial\Omega_R \\ u > 0 &\quad \text{in} \quad \Omega_R, \end{aligned} \tag{P}$$

where $\Omega_R \equiv \{x \in \mathbb{R}^N | R-1 < |x| < R+1\}$ and $1 < p < (N+2)/(N-2)$ for $N \geq 3, 1 < p < \infty$ for $N = 2$.

The problem (P) is invariant under the orthogonal coordinate transformation, that is, $O(N)$-symmetric. When the domain Ω_R is a ball. the solutions are $O(N)$-symmetric, in other words, radially symmetric. This is an elegant result of Gidas, Ni and Nirenberg [GNN]. On the basis of the symmetric property of the solutions, a uniqueness of a solution of problem (P) was proved [GNN],[NN]. On the other hand, although an annulus has the same symmetric property with a ball, Brezis and Nirenberg pointed out in [BN] that there exists a non-radial solution of problem (P) when $R > 1$, $n \geq 3$ and $(N+2)/(N-2) - p$ is positive and sufficiently small. In fact, they showed that the minimal energy solutions of problem (P) is not radial symmetric in the above case. Furthermore, Coffman [Co] proved that, in two-dimensional case, the number of non-radial and nonequivalent solutions of problem (P) goes to ∞ as $R \to \infty$. Here we say that u and v in $H_0^{1,2}(\Omega_R)$ are nonequivalent if $u(\cdot) \neq v(g\cdot)$ for any $g \in O(N)$. The same result was obtained by Y.Y. Li for $N \geq 4$ [Li] and by the author for $N = 3$ [By1]. In [BN], [Co], [Li], [Lin]

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and [MS], the non-radial solutions of (P) which have globally minimal energies in some symmetric functions classes have been studied. On the other hand, in [By1] the author proved an existence of locally -rather than globally- minimal energy solutions of (P) in certain symmetric functions classes when the space dimension is three, from which it was shown that the number of nonequivalent non-radial positive solutions of (P) goes to infinity as $R \to \infty$.

It is interesting to note that the $O(N)$-symmetry has two contrasting effects on the structure of the positive solutions: when the domain is a ball, the symmetry makes the structure of solutions of problem (P) to be simple; on the other hand, when the domain is an annulus, to be complicated. Thus, it is natural to wonder why this contrasting effect of the $O(N)-$ symmetry on the structure of the positive solutions occurs. In this article, it will be illustrated that the contrasting effect of the symmetry is due to the contrasting structure of the orbits space of group actions $G \times \Omega_R \to \Omega_R, G \subset O(N)$, for each cases $R > 1$ and $R < 1$. In fact, we will see that a *critical orbital set* produces a corresponding solution of our problem. When $R < 1$, there is only one *critical orbital set*, $\{0\}$. On the other hand, when $R > 1$, there are many interesting *critical orbital set*; thus a rich structure of the orbits space brings about a rich variety of positive solutions of (P) as $R \to \infty$, almost all of which never have been found in the literature.

1. Main results

Let G be a closed subgroup of $O(N)$. We denote

$$H_R^G \equiv \{u \in H_0^{1,2}(\Omega_R) \mid u(g\cdot) = u(\cdot),\ g \in G\}.$$

For any $u \in H_0^{1,2}(\Omega_R)$, we define its energy

$$\Gamma(u) \equiv \frac{1}{2}\int_{\Omega_R} |\nabla u|^2 - \frac{1}{p+1}\int_{\Omega_R} u^{p+1} dx.$$

Let Γ^∞ be the minimal energy of the solutions of

$$\begin{aligned} \Delta u + u^p = 0 &\quad \text{in } (-1,1) \times \mathbb{R}^{N-1} \\ u > 0 &\quad \text{in } (-1,1) \times \mathbb{R}^{N-1} \\ u = 0 &\quad \text{on } \{-1,1\} \times \mathbb{R}^{N-1}. \end{aligned}$$

For a closed subgroup $G \subset O(N)$, we consider a natural group action $G \times S^{N-1} \to S^{N-1}$. We denote the $G-$orbit of x by $xG \equiv \{g \cdot x|\ g \in G\}$. Then, the orbit xG is a closed submanifold of S^{N-1}. Thus we can define $\mathbf{d}(xG)$ the dimension of the orbit xG. Let $\mathbf{m}(xG)$ be the $\mathbf{d}(xG)-$dimensional Housdorff measure. We, then, give a partial order $\prec\prec$ on the space $\{xG | x \in S^{N-1}\}$ as follows:

$$xG \prec\prec yG$$

if and only if

$$\mathbf{d}(xG) < \mathbf{d}(yG),$$

or

$$\mathbf{d}(xG) = \mathbf{d}(yG) = 0 \text{ and } \mathbf{m}(xG) < \mathbf{m}(yG).$$

Then we define a locally minimal orbital set under the action $G \subset O(N)$ as follows.

Definition. Let G be a closed subgroup of $O(N)$. A set $M \subset S^{N-1}$ is called a locally minimal orbital set under the action of G if M is invariant under the action of G and a minimal set satisfying the following conditions:

(i) for any $x, y \in M$, it holds that $\mathbf{d}(xG) = \mathbf{d}(yG)$, and that $\mathbf{m}(xG) = \mathbf{m}(yG)$ when $\mathbf{d}(xG) = \mathbf{d}(yG) = 0$.

(ii) there exists a positive constant $\delta > 0$ such that for any $y \in \{z \in S^{N-1} | \operatorname{dist}(z, M) \leq \delta\} \setminus M$ and $x \in M$, it holds that $xG \prec\prec yG$.

A locally minimal orbital set M under the action of G is called a finite type, or an infinite type if $\mathbf{d}(xG) = 0$, or $\mathbf{d}(xG) > 0$ for any $x \in M$, respectively.

We have the following properties of a locally minimal orbital set.

Proposition [By2-3]. *A locally minimal orbital set is a smooth manifold. Moreover, if a locally minimal orbital set is of a finite type, its connected components are totally geodesic.*

For any closed subgroup G of $O(N)$, we define

$$H_R^G \equiv \{u \in H_0^{1,2}(\Omega_R) | u(x) = u(gx) \text{ for any } x \in \Omega_R, g \in G\}.$$

Then we have the following results.

Theorem 1 [By2]. *Let M be a locally minimal orbital set of a finite type under the action of $G \subset O(N)$. Then, for sufficiently large $R > 0$, there exists a solution $u_R \in H_R^G$ of problem (P) such that*

(i) *for some $x_R \in \{Rx | x \in M\}$ and $C, c > 0$, independent of R,*

$$u_R(x) \leq C \exp(-c\,dist(x, x_R G));$$

(ii) *if x_R is a maximum point of u_R, then $x_R/|x_R| \in M$; and*

(iii) *for any $x \in M$,*

$$\lim_{R \to \infty} \Gamma_R(u_R) = \mathbf{m}(xG)\Gamma^\infty.$$

Theorem 2 [By3]. *Let M be a locally minimal orbital set of an infinite type under the action of $G \subset O(N)$. For sufficiently large $R > 0$, there exists a solution $u_R \in H_R^G$ of problem (P) such that*

(i) *if y_R is a maximum point of u_R, then $y_R/|y_R| \in M$ for large R;*

(ii) *there exist constants $C, c > 0$, independent of R, such that*

$$u_R(x) \leq C \exp(-c\ dist(x, y_R G));$$

and

(iii) *there exists a constant $C > 0$, independent of R, such that for any $x \in M$,*

$$1/C \leq \Gamma(u_R)/R^{\mathbf{d}(xG)} \leq C.$$

Theorem 3 [By4]. *Let $M^1, \cdots, M^k$ be disjoint locally minimal orbital sets of finite types under the action of G. Then, for sufficiently large $R > 0$, there exists a solution $u_R \in H_R^G$ of problem (P) with the following properties:*

(i) *the u_R has local maximum points $x_R^i \in \{x \in \Omega_R | x/|x| \in M^i\}, i = 1, \cdots, k$, such that for some constants $C, c > 0$, independent of R,*

$$u_R(x) \leq C \exp(-c \min_{\{i=1,\cdots,k\}} dist(x, x_R^i G));$$

(ii) *if $y_R \in S^{N-1}$ is a local maximum point of*

$$\xi_R(y) \equiv \max\{u_R(sy) | s \in (R-1, R+1)\},\ y \in S^{N-1},$$

then $y_R \in \cup_{i=1}^k M^i$; and

(iii) *for any $x^i \in M^i, i = 1, \cdots, k$,*

$$\lim_{R\to\infty} \Gamma_R(u_R) = \sum_{i=1}^k \mathbf{m}(x^i G)\Gamma^\infty.$$

2. Concluding remarks

For a closed subgroup $G \subset O(N)$, let $M^1, \cdots, M^k$ be disjoint locally minimal orbital sets of infinite types under the action of G. Like as in Theorem 3, it is expected that for sufficiently large $R > 0$, there exists a solution $u_R \in H_R^G$ of problem (P) such that

(i) the u_R has local maximum points $x_R^i \in \{x \in \Omega_R | x/|x| \in M^i\}, i = 1, \cdots, k$, such that for some constants $C, c > 0$, independent of R,

$$u_R(x) \leq C \exp(-c \min_{\{i=1,\cdots,k\}} \mathrm{dist}(x, x_R^i G));$$

(ii) if $y_R \in S^{N-1}$ is a local maximum point of

$$\xi_R(y) \equiv \max\{u_R(sy) | s \in (R-1, R+1)\},\ y \in S^{N-1},$$

then $y_R \in \cup_{i=1}^k M^i$.

For a closed subgroup of $G \subset O(N)$, let Υ_G be the set of locally minimal orbital sets of finite types under the action of G. Theorem 3 says that for any $M^1, \cdots, M^k \in \Upsilon_G$ and large $R > 0$, there exists a solution u_R of problem (2) such that the local maximum points of $\xi_R(y) \equiv \max\{u_R(sy) | s \in (R-1, R+1)\}, y \in S^{N-1}$, is exactly same with the set $\cup_{i=1}^k y_i G$ for some $y_i \in M^i, i = 1, \cdots, k$. Moreover, the function ξ_R satisfies the following properties:

$$0 < \liminf_{R\to\infty} \xi_R(y_i) \leq \limsup_{R\to\infty} \xi_R(y_i) < \infty,\ i = 1, \cdots, k$$

and

$$\lim_{R\to\infty} \xi_R(x) = 0 \text{ for } x \in S^{N-1} \setminus \cup_{i=1}^k y_i G.$$

Thus, the function ξ_R on S^{N-1} exhibits symmetric ($G-$invariant) spikes as $R \to \infty$. Furthermore, it is anticipated that each solution of problem (2) has a certain symmetric property. More precisely, let v_R be a solution of problem (2) such that $\limsup_{R\to\infty} \Gamma_R(v_R) < \infty$. Then, it seems that for large $R > 0$, there exists a closed subgroup G_R of $O(N)$ with the following property: there exist $M_R^1, \cdots, M_R^l \in \Upsilon_{G_R}$ such that the local maximum points of $\zeta_R(y) \equiv \max\{v_R(sy) | \; s \in (R-1, R+1)\}, y \in S^{N-1}$, is exactly same with $\cup_{i=1}^l y_R^i G_R$ for some $y_R^i \in M_R^i, i = 1, \cdots, l$. Typically, we consider solutions w_R of problem (2) such that $\limsup_{R\to\infty} \Gamma_R(w_R) \le 2\Gamma^\infty$. Then, we can show that for large $R > 0$, there are at most two local maximum points of $\eta_R(y) \equiv \max\{w_R(sy) | \; s \in (R-1, R+1)\}, y \in S^{N-1}$. If there exists only one local maximum point y_R of η_R, then, by a similar method as in the proof of [By1, Proposition 2.4], we can show that $w_R \in H_R^{G_R}$ for $G_R \equiv \{g \in O(N) | \; g \cdot y_R = y_R\}$, that is, the w_R is $O(N-1)-$symmetric. If there are two local maximum ponts $y_R^1, y_R^2 \in S^{N-1}$ of η_R, we expect that $y_R^1 + y_R^2 = 0$ for large $R > 0$.

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