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METHODS OF PROVING SYMMETRY OF SOLUTIONS TO ELLIPTIC BOUNDARY VALUE PROBLEMS

JEONG-SEON BAEK

Abstract. This paper is a brief survey on the method of proving symmetry of solutions to semilinear elliptic problems involving Laplacian, p-Laplacian, poly-Laplacian, A-Laplacian and convex-Laplacian operators. Various results are introduced and a brief illustration on the most successful method, the moving plane method, is shown.

1. Model Problem

Numerous authors have given substantial attention to the following problem. Under what additional conditions, are the solutions of

$$-\Delta u = f(x,u) \qquad \Omega \subset \mathbb{R}^n \tag{1}$$

radially symmetric?

2. Non-radial Example

Let Ω be the unit ball, and λ_k be the k-th eigenvalue of the Laplacian with $k \geq 2$ such that the corresponding eigenvalue $w(x)$ is not radially symmetric. Let $f(x,u) = 2n + \lambda_k(|x|^2 - 1 + u)$. Then $u(x) = 1 - |x|^2 + \epsilon w(x)$ is a solution of (1) with the Dirichlet vanishing boundary condition. If ϵ is small enough, then $u(x) > 0$ in Ω but $u(x)$ is not radially symmetric.

3. Typical affirmative results

1. Assume that $f(x,u) = g(|x|,u)$ is locally Lipschitz in the variable u is non-increasing in $|x|$ and Ω is a ball. If $u(x)$ is a positive solution of (1) with vanishing boundary data, then $u(x)$ is radially symmetric. (See [17])
2. If some level set of a solution $u(x)$ is symmetric around the origin then $u(x)$ is radially symmetric. (Application of 1.)
3. If $f(x,u) = g(|x|^2,u)$ is C^1 and $f_u < \lambda_2$, then every solution $u(x)$ with vanishing boundary condition on a symmetric domain Ω is symmetric. Here λ_2 is the second eigenvalue of Dirichlet problem for the Laplacian. (By direct computation one can show that $v_{ij} = x_i u_j - x_j u_i$ is orthogonal to the first eigenfunction. From the energy comparison, it follows that $v_{ij} = 0$. So u is symmetric.)

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4. If $f(x,u)=1$, Ω is a C^1 domain, $u(x)=0$ and $|\nabla u(x)|$ is constant on $\partial\Omega$, then Ω is a ball and $u(x)$ is radially symmetric. (See [31]. One can prove this result using integral identities in one page. The crucial step is to show that two sides of the inequality $\sum u_{ij}^2 \geq (\Delta u)^2/n$ is equal.) Note that this result is equivalent to a converse of the mean value theorem:

   Corollary. *A C^1 domain Ω is a ball if*

$$\frac{1}{|\Omega|}\int_\Omega h = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} h$$

   for all h continuous in $\bar{\Omega}$ and harmonic in Ω.
5. Assume that $f(x,u)=g(x)$ and let g^* be the Schwartz symmetrization of g. Let $v=(-\Delta)^{-1}g^*$. If the solution $u^*(0)=v(0)$, then $u(x)$ is radially symmetric. (See [4,5,6])
6. Let Ω be the exterior of a C^2 Jordan Domain in $\mathbb{R}^n$, $n \geq 3$. Assume that $f(x,u)=g(u)$ is nonnegative and $u^{\frac{-n+2}{n-2}}g(u)$ is non-increasing. If $u(x)$ is a solution satisfying $u(x)=a$ and $|Du|=c$ on $\partial\Omega$ and $0 \leq u(x) \leq a$ in Ω with some constants a and c, then Ω is the exterior of a ball and $u(x)$ is radially symmetric and strictly decreasing in any radial direction. (See [1,2])
7. (See [29].) Let $u(x)$ be a solution of

$$\Delta u + f(u,|\nabla u|) = 0 \qquad \text{in } \mathbb{R}^n \setminus \bar{\Omega},$$

$$u = a \qquad \text{on } \partial\Omega,$$

$$\frac{\partial u}{\partial \nu} = c \qquad \text{on } \partial\Omega,$$

$$u = 0, \quad |\nabla u| = 0 \qquad \text{at } \infty,$$

$$0 \leq u < a \qquad \text{in } \mathbb{R}^n \setminus \Omega,$$

   where a and c are constants, and $n \geq 2$. Assume that Ω is a Jordan domain and either

   (H_1): $f(t,s)=f_1(t,s)+f_2(t)$ with f_1 Lipschitz continuous f_2 nondecreasing in t and f non-increasing in t for small positive values of t and s, or

   (H_2): $f(t,s)$ is Lipschitz continuous and non-increasing in t.

   Then Ω is a ball, and $u(x)$ is radial and $u' < 0$.
8. In the previous statement, assume Ω be the exterior of a ball. Then without the Neumann boundary condition, the same conclusion holds.
9. Let $\Omega = \mathbb{R}^n$. If $f(x,u)=g(u)$ is $C^{1+\alpha}$ for u near 0, $g(0)=0$ and $g'(0)<0$, then $u(x)>0$ satisfying $u(\infty)=0$ is radially symmetric.
10. $f(x,u)=g(|x|,u)=O(|x|^{-m}), m>0$ near ∞.
    (a) for $r \geq 0$ and $0 < t \leq u_0 = \max u$, $g(r,t)$ is continuous, positive, nondecreasing in s and strictly increasing in r.
    (b) for some $p > (n+1)/m$ and some constant $C>0$, $g(r,u) \leq Cu^p$ for $u \leq u_0$. Then any positive solution $u(x)$ is radially symmetric about the origin and $u'(r)<0$. (See [18,24,25])
11. ([Scalar Curvature equation]) If $f(x,u)=\frac{1}{1+|x|^\tau}u^{(n+2)/(n-2)}$ with $\tau > 2$, then any bounded positive solution $u(x)$ is radially symmetric.
12. ([Matukuma equation]) If $f(x,u)=\frac{u^p}{1+|x|^2}$ with $p>1$, then any bounded positive solution $u(x)$ with $\int_{\mathbb{R}^n}\frac{u(x)^p}{1+|x|^2}dx < \infty$ must be radially symmetric.

4. p-Laplacian Case

For the problem

$$\Delta_p u = f(u) \in \quad \text{in } \mathbb{R}^n, \quad u > 0,$$

1. ([Damascelli and Pacella])
   (a) f is locally Lipschitz continuous in $(0, \infty)$.
   (b) There is $s_0 > 0$ such that f is non-increasing on $(0, s_0)$.
   If $1 < p < 2$ and $u(x) > 0$ is a weak solution, then $u(x)$ is radially symmetric and $u'(r) < 0$.
2. Elena Sartori's result: Let $\Omega = \Omega_0 \setminus \bar{\Omega}_1$ be a difference of domains $\Omega_1 \subset \Omega_0$ star-like around the origin. Consider that free boundary problem

$$\Delta_p u = 0 \qquad \text{in } \Omega$$

$$u \to 1, \quad |\nabla u| \to c_1 \qquad \text{uniformly a.e. as } x \to \partial\Omega_1$$

$$u \to 0, \quad |\nabla u| \to c_0 \qquad \text{uniformly a.e. as } x \to \partial\Omega_0$$

5. A-Laplacian Case(See [32])

Consider the problem of the form

$$\operatorname{div}(A(|\nabla u|\nabla u) + f(u) = 0, \qquad x \in \mathbb{R}^n, \quad n \geq 2, \tag{5}$$

under the ground state condition

$$u(x) \to 0 \quad \text{as} \quad |x| \to \infty \tag{6}$$

Let $B(t) = tA(t)$ and $F(t) = \int_0^t f(s)ds$. Assume that
   (a) $f(t)$ is continuous for $t \geq 0$ and Lipschitz continuous for $t > 0$, and
   (b) $f(0) = 0$ and f is non-increasing on some interval $[0, \delta]$.

1. If $f \in C^{1,1}([0, \infty))$ and $B'(t) > 0$ for all $t \geq 0$, then $u(x) \geq 0$ whose (open) support is connected must be radially symmetric, and the $u'(r) < 0$ for all $r > 0$ such that $u(r) > 0$.
2. If $f(t)$ is locally Hölder continuous for $t \geq 0$ and satisfies

$$\int_0 \frac{dt}{|F(t)|^{1/2}} < \infty$$

   then $u(x) \geq 0$ has (open) support on a finite number of open balls in $\mathbb{R}^n$, on each of which it is radially symmetric about the center of the ball.
3. Assume that $f \in C^{1,1}((0, \infty))$ and $B'(t) > 0$ and $B(0+) = 0$. If the solution $u(x) \geq 0$ satisfies that

$$\{x \in \mathbb{R}^n \mid u(x) > 0, \quad \nabla u(x) = 0\}$$

   contains exactly one point, then $u(x)$ is radially symmetric and $u'(r) < 0$ whenever $u(r) > 0$.
4. Assume that $f \in C^{1,1}((0, \infty))$ and $B'(t) > 0$ and $B(0+) = \infty$. If $u(x) \geq 0$ with the property that the (open) support of ∇u is connected, then $u(x)$ is radially symmetric and $u'(r) < 0$.

6. Convex-Laplacian

Let $\alpha(\xi)$ be a convex function of degree 1, and let $\div A(-\nabla u)$, called a convex Laplacian, be the Euler-Lagrange derivative of the functional $\int \alpha(-\nabla u(x))^p dx$. If $\alpha(-\xi) = \alpha(\xi)$, we call it is balanced. Some authors studies convex Laplacian([3,6]) by symmetrization method. By abuse of language, the symmetric solutions are of the form $u = u(\alpha^\circ(x))$ in the balanced case and $u = u(\alpha^\circ(x))$ or $u(\alpha^\circ(-x))$ according as $u' < 0$ or $u' > 0$ in the unbalanced case. Here α° is the polar or the dual of α.

7. Poly-Laplacian Case(Higher Order Laplacian)

1. ([Dalmasso]) Consider the problem
$$(-\Delta)^m u = f(|x|, u) \qquad \text{in } B$$
$$u = \frac{\partial u}{\partial \nu} = \cdots = \left(\frac{\partial}{\partial \nu}\right)^{m-1} u = 0 \qquad \text{on } \partial B$$
where B is the unit ball of $\mathbb{R}^n$. If $f(r,u)$ is C^1 and $f_u \le \lambda_2$, then any solution $u(x)$ is radially symmetric.
2. ([26,37]) Let $u(x)$ be a solution of
$$(-\Delta)^m u = (n-1)! e^{nu} \quad \text{in } \mathbb{R}^n, n = 2m$$
satisfying
$$\int_{\mathbb{R}^n} e^{nu} < \infty, \quad u(x) = o(|x|^2) \text{ at } \infty$$
Then
$$u(x) = \log \frac{2\lambda}{\lambda^2 + |x - x_0|^2}$$
with some x_0 and $\lambda > 0$.
3. ([26,37]) Let $u(x)$ be a positive solution of
$$(-\Delta)^m u = u^{\frac{n+2m}{n-2m}} \qquad \text{in } \mathbb{R}^n, n > 2m > 0.$$
Then
$$u(x) = \left(\frac{2\lambda}{1 + \lambda^2 |x - x_0|^2}\right)^{\frac{n-2m}{2}}$$
with some x_0 and $\lambda > 0$.

8. Briefing of Methods

The most successful method of proving symmetry is the **moving plane method** developed by Alexandrov and introduced in PDE by Serrin[31]. The most important contributions are done by Gidas, Ni and Nirenberg[17,18] and by Berestycki and Nirenberg[7]. Many authors used this method to get symmetry results.

Let $e \in \mathbb{R}^n$, $|e| = 1$, $\lambda \ge 0$ and $u_\lambda(x) = u(x_\lambda)$, where $x_\lambda = x - 2(x \cdot e - \lambda)e$ is the reflection of x with respect to the hyper-plane $\{y \mid y \cdot e = \lambda\}$. The moving plane method is to prove that under some conditions the set $\mathcal{I} = \{\lambda \mid u_\lambda(x) \ge u(x) \text{ for } x \cdot e \ge \lambda\}$ is connected and nonempty.

For the illustration of the method let's consider the simplest case when $f(t)$ is Lipschitz continuous. Then every positive weak solution of the problem
$$-\Delta u = f(u)$$

in the unit ball of $\mathbb{R}^n$ with the vanishing boundary condition is symmetric with respect to the center of the ball. The clue step of the method is the following lemma and the Hopf boundary point lemma.

Lemma *Let $a(x)$ be a bounded function and $w \in H_0^1(\Omega)$ be a weak solution to the problem*

$$-\Delta w = aw.$$

Assume that $|\{x \mid w(x) > 0\}| > 0$. Then $|\{x \mid w(x) > 0\}| \geq C$, where C is independent of w.

Proof. Test the equation with w^+ and apply the Sobolev inequality and Hölder inequality, then one can get more general result.

In the following we consider functions on $\Omega_\lambda = \{x \mid x \cdot e > \lambda\}$. For λ near to 1, $w = (u_\lambda - u)^-$ must be vanishing by this lemma. Hence $\mathcal{I}$ is not empty. Closedness of $\mathcal{I}$ is obvious. To show that $\mathcal{I}$ is open, let $\lambda \in \mathcal{I}$ and $0 < \lambda < 1$. By Hopf boundary point lemma, $\nabla(u_\lambda - u) \cdot e > 0$ on the hyper-plane portion of the boundary of Ω_λ. Hence $\{x \mid w(x) > 0\}$ lies near to the unit circle as λ moves slightly smaller. The above lemma implies $w = 0$. Thus $\mathcal{I}$ is open in $[0, 1]$. We conclude that $\mathcal{I} = [0, 1]$.

Another interesting method is a symmetrization method consisting of applications of some isoperimetric inequalities and comparison principles via symmetrization. In rather general cases, variational ground solutions are proved to be symmetric by this method.

Yet another one is the shape deformation method or the domain derivative method. For specific or general works using these methods are listed in the references.

□

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Department of Mathematics, Chonnam National University, Kwangju, Korea
E-mail address: jsbaek@chonnam.chonnam.ac.kr