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EXISTENCE AND MULTIPLICITY OF PERIODIC SOLUTIONS FOR NONLINEAR TELEGRAPH EQUATIONS*

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ABSTRACT. The existence and multiplicity of nonlonear telegraph equations will be discussed

1. INTRODUCTION

In this article, we study the recent developement of the exixtence and multiplicity results for nonlinear telegraph equations of the following form which has nonlinear internal-energy term

$$u_{tt} - \Delta u + \beta u_t + g(t,x,u) = h(t,x), \ (t,x) \in R^2,$$

where $\beta \neq 0$, $u = u(t,x)$ and g is a continuous function. This equation is related to the mathematical interpretation of undisorted plane waves. Indeed, for the wave equation $\frac{1}{c^2}u_{tt} = \Delta u$, progressing undistored plane waves with the speed c and the arbitrary form

$$\phi(\sum_{i=1}^{n} \alpha_i x_i - ct), \ \sum_{i=1}^{n} \alpha_i^2 = 1$$

are possible in every direction. A more more general interpretation is given by the telegraph equation

$$u_{tt} - c^2 \Delta_x u + (\alpha + \beta) u_t + \alpha\beta u = 0$$

satisfied by the voltage or the current u as a function of time t and the position x along a cable, here x measures the length of the cable from an initial point. This differential equation is derived by elimination of one of the unknown functions from the following system of differential equations of first order for the current i and the voltage u as a function x and t:

$$Cu_t + Gu + i_x = 0.$$

$$Li_t + Ri + u_x = 0.$$

Here L is the inductance of the cable, R its resistance, C its shunt capacity, and, finally, G its shunt conductance(loss of current divided by voltage). The constants in telegraph equation, which arise in the elimination process, we have the meaning

$$\frac{1}{c^2} = LC, \quad \alpha = \frac{G}{C}, \ \beta = RL,$$

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where c is the speed of light and α the capacitive and β the inductive damping factor.

We will survey the existence results then we discuss multiplicity results and our method of proof on multiplicity results is mainly based on Topological degree.

2. Main results

Existence Results

Mawhin[M1] proves the existence of 2π-periodic solutions in both variables to the equation of the form:

$$u_{tt} - u_{xx} + \beta u_t + g(u) = h(t, x),\ \ (t, x) \in R^2$$

where $\beta \neq 0$, $u = u(t, x)$ and g is a continuous function on R of at most linear growth satisfying one of the following conditions

(A) $\gamma \leq u^{-1}g(u) \leq \delta$ for all sufficiently large $|u|$ with $\gamma, \delta \in (\mu_1, \mu_2)$,

(B) $\lim_{u\to-\infty} u^{-1}g(u) = \gamma$, $\lim_{u\to\infty} u^{-1}g(u) = \delta$ and $\gamma, \delta \in [\mu_1, \mu_2], \gamma \neq \delta$,

(C) $\lim_{|u|\to\infty} u_{-1}g(u) = \mu_1, \mu_1$ is not a perfect square,

(D) $\lim_{|u|\to\infty} u^{-1}g(u) = q^2$ for some $q \in Z$ provided that β and g satisfy certain additional requirement.

Here μ_1 and μ_2 are consective elements of the set $\sum = \{k^2 - j^2 | k, j \in Z\} = \{Z+1\} \cup 4Z$. The author utlizes the compactness properties and the degree theory.

Mawhin[M1] and Fucik and Mawhin[F1] deal with 2π-periodic solutions(in both variables) to the equations of the form:

$$u_{tt} - u_{xx} + \beta u_t = \mu u^+ + \nu u^- + g(u) = h(t, x),\ \ (t, x) \in R^2,$$

where β, μ, $\nu \in R$, $\beta \neq 0$, $u = u(t, x)$ and g is a continuous bounded function. The authors decompose the set R^2 of all pairs (μ, ν) (see [S1]) and obtain existence results of different types on different compont.

Kim[k1] proves the existence of weak solution of the Dirichle-periodic problem on $(0, 2\pi) \times (0, 2\pi)$ for the equations of the form:

$$u_{tt} - u_{xx} + \beta u_t + g(u) = h(t, x),\ \ (t, x) \in (0, 2\pi) \times (0, 2\pi)$$

When $\beta \neq 0$ and $u = u(t, x)$, he proves the existence of a weak solution of the equation for all $h \in L^2$ under the following two assumptions:

(E) The growth condition $|g(u)| \leq |u|^p + b$ holds for $u \in R$, where $a > 0, b \leq 0$ and $p \geq 1$ are constant,

(F) There exist numbers α, τ such that the inequality $ug(u) \geq -\alpha|u| - \tau$ holds for $u \in R$.

Kim[K2] studies the problem of existence of weak solutions to the double-periodic problem for equations of the form:

$$u_{tt} - u_{xx} + \beta u_t - g(t, x, u) = h(t, x),\ \ (t, x) \in (0, 2\pi) \times (0, 2\pi)$$

where$\beta \neq 0$ and $u = u(t, x)$. The nonlinear term g is assumed to be a Caratheodory function that is decreasing in u and satisfies some restriction at infinity. In the proof, a continuation theorem of Mawhin is used, along with a convolution product representation of the linear part of the above equation, that provides a essential norm estimate.

Kim[K3] discusses the existence of weak solutions to the Dirichlet-periodic problem for the equations of the form:

$$u_{tt} - u_{xx} + \beta u_t - u - g(t,x,u) = h(t,x),\ (t,x) \in (0,2\pi) \times (0,\pi)$$

where $\beta \neq 0$, $u = u(t,x)$ and g is a Caratheodory function satisfying suitable conditions, and h satisfies Landesman-Lazer-like conditions. the key to the proof lies in finding a priori bounds for solutions,by using appropriate projections and a bootstrap argument.

Hirano, Kim[H1] syudy the existence of weak solutions to the Dirichlet-periodic problem for the equation of the form:

$$u_{tt} - u_{xx} + \beta u_t - u - g(t,x,u) = h(t,x),\ (t,x) \in (0,2\pi) \times (0,\pi)$$

where $\beta \neq 0$, $u = u(t,x)$. Imposing no boundedness on the nonlinear term g. They show that the problem has at least one weak solution. Their proof is based on the Leray-Schauder continuation theorem.

Let $\Omega \subseteq R^n$ be bounded domain with smooth boundary $\partial\Omega$. Brezis and Nirenberg[B2] apply their general results on the ranges of nonlinear operators, for example, to the Dirichlet problem for the equations of the form:

$$u_{tt} + Eu + \beta u_t + g(t,x,u) = h(t,x),\ (t,x) \in R \times \Omega,$$

where $\beta \neq 0$, $u = u(t,x)$ and

$$Ev(x) = \sum_{|\alpha|,|\beta| \leq m} (-1)^{|\alpha|} D^{\alpha}(a_{\alpha\beta}(x) D^{\beta} v(x)).$$

The function g is supposed to be continuous in u and to satisfy

(G)$|g(t,x,u)| \leq \gamma|u| + b(t,x)$ (for some $\gamma > 0$, $b \in L^2$) and for some c_0

$ug(t,x,u) \leq \delta(t,x)|u| - d(t,x)$(for some $\delta \in L^2$,$d \in L^1$),

$\int_0^{\omega}\int_{\Omega} g_+ v^+ dxdt - \int_0^{\omega}\int_{\Omega} g_- v^- dxdt \geq \int_8^{\omega}\int_{\Omega} hvdxdt + c_0(\int_0^{\omega}\int_{\Omega}|v|^2 dxdt)^{1/2}$ for all $v \in Ker(E)$.

Here $g_+(t,x) = \lim_{u\to\infty} \inf g(t,x,u)$, $g_-(t,x) = \lim_{u\to-\infty} \inf g(t,x,u)$, $v^+ = \max\{v,0\}$ and $v^- = \max\{-v,0\}$.

The authors prove the existence of a generalized 2π-periodic solution, provided that one of the conditions (H) and (I) is satisfied:

(H) (G) holds for every $\gamma > 0$(with $b = b_\gamma$),

(I) $\beta_- \leq \beta_+$ and (e) holds with $\gamma < \alpha$, where $\alpha = \inf\{|j^2 - \lambda_k + \frac{a^2 j^2}{j^2 - \lambda_k}||j,k \in Z, \lambda_k < j^2\}$ and the λ_k are eigenvalues of E).

The regularity results are also stated

Biroli[B1] treats the Dirichlet problem for the equation of the form

$$u_{tt} - \Delta_n u + \beta u_t + g(u) = h(t,x),\ (t,x) \in R \times \Omega,$$

where $\beta > 0$ $u = (t,x)$ and g is a monotone function on R, $g(0) = 0$. He proves the existence of a solution u with regularity.

Horacek[H2],[H3] assert(without proof) that the Dirichlet problem for the equation of the form:

$$u_{tt} - \Delta_n u + u_t + u^3 = h(t,x),\ (t,x) \in R \times \Omega,$$

where $\beta \neq 0$ and $u = (t,x)$, has a generalized or a calssical ω-periodc solution provied that $n = 2$ or $n = 1$, respectively

Multiplicity Results

Kim[K4] deal with the weakened Ambrosetti-Prodi(WAP) type and Ambrosetti-Prodi(AP) type multiplicity results for weak doubly periodic solutions to nonlinear dissipative hyperbolic equations. As a AP-type multiplicity result, he shows that there exists a real mumber s_1 such that the equation of the form:

$$u_{tt} - u_{xx} + \beta u_t + g(u) = h(t,x),\ (t,x) \in (0,2\pi) \times (0,2\pi)$$

where $\beta \neq 0$ and $u = u(t,x)$. has non, at least one, or at least two weak doubly periodic solutions for $s < s_1$, $s = s_1$ or $s > s_1$ respectively.

Kim[K5] considers the nonlinear equations of the form:

$$u_{tt} - u_{xx} + \beta u_t + g(t,x,u) = h(t,x),\ (t,x) \in (0,2\pi) \times (0,2\pi)$$

with 2π-periodicity conditions with respect to both x and t, where $\beta \neq 0$ constant, $u = u(t,x)$, h is given function and g is continuous with linear growth in u such that $g(\cdot,\cdot,u) \to \infty$ uniformly as $|u| \to \infty$. Assuming furthermore that g is nonnegative and dominated by h in a suitable sense, he shoes the existence of at least two distinct solutions. Analogous results are obtained in the nonnegativity condition is replaced by the assumption that g does not depend on x and t and is Lipschitz with a sufficiently small coefficient.

Let $\Omega \subseteq R^n,\ n \geq 1$, be a bounded domain with smooth boundary $\partial\Omega$ which is assumed to be of class C^2 and let $Q = (0,2\pi) \times \Omega$. Kim[K6] investigate the Ambrosetti - Prodi type multiplicity result for weak Dirichlet-periodic solution of the nonlinear equations of the form:

$$u_{tt} - \triangle_x u + \beta u_t - \lambda_1 u + g(u) = \frac{s\phi_1}{\sqrt{2\pi}} + h(t,x) \quad \text{in} \quad Q$$

where $\beta \neq 0$, $u = u(t,x)$, g is continuous function with at most linear growth in u and λ_1 denotes the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition and ϕ_1 is the correspanding positive normalized eigenfunction; i.e., $\phi_1(x) > 0$ on Ω and $\int_\Omega \phi_1^2(x)dx = 1$, and $h \in L^2(Q)$ with

$$\iint_Q h(t,x)\phi_1(x)dtdx = 0.$$

He prove the multiplicity result under the following assumptions:

(J)

$$\lim_{|u|\to\infty} \inf g(u) = +\infty,$$

(K)

$$\lim_{u \to -\infty} \sup |\frac{g(u)}{u}| < \lambda_2 - \lambda_1,$$

where λ_2 is the second eigenvalue of of $-\Delta$ with zero Dirichlet boundary data and

Kim[K7] comsider the multiplicity for Dirichlet-periodic solution of the equations of the form

$$u_{tt} - \triangle_x u - \lambda_1 u + \beta u_t + g(u) = h(t,x) \text{ in } \ Q,$$

where $\beta \neq 0$, $u = u(t,x)$.

He prove his multiplicity result which is different from Ambrosetti-Prodi type under the conditions (J) and (K) and this result extent one of the rersults in [K5] and the proof is based on topological degree.

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