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SELF-SIMILAR SOLUTIONS OF A PARABOLIC PARTIAL DIFFERENTIAL EQUATION

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Abstract. In this note we classify positive solutions of an equation

$$\Delta u + \frac{1}{2} x \cdot \nabla u + \frac{1}{p-1} u - |u|^{p-1} u = 0 \qquad \text{in} \qquad \mathbf{R}^N,$$

where $1 < p < (N+2)/N$. We announce both what is done recently and what should be done sooner or later.

1. Introduction

In this note we investigate some properties of positive solutions of an equation

$$\Delta u + \frac{1}{2} x \cdot \nabla u + \frac{1}{p-1} u - |u|^{p-1} u = 0 \qquad \text{in} \qquad \mathbf{R}^N, \tag{1}$$

where $1 < p < (N+2)/N$. This equation arises naturally in the study of the asymptotic behavior of solutions of a semilinear parabolic equation

$$v_t = \Delta v - |v|^{p-1} v \qquad \text{in} \qquad \mathbf{R}^N \times (0, \infty). \tag{2}$$

We first observe that if $v(x,t)$ solves (2), then for $\lambda > 0$, the rescaled functions

$$v_\lambda(x,t) = \lambda^{2/(p-1)} v(\lambda x, \lambda^2 t) \tag{3}$$

defines a one parameter family of solutions to (2). A solution v is said to be self-similar when $v_\lambda(x,t) = v(x,t)$ for every $\lambda > 0$. It can be easily verified that v is a self-similar solution to (2) if and only if v has the form

$$v(x,t) = t^{-1/(p-1)} u(x/\sqrt{t}), \tag{4}$$

where u satisfies (1).

The asymptotic behavior of solutions of (2) is usually determined by the limiting profile of (3) as $\lambda \to \infty$, which becomes a self-similar solution (see [3]). Henceforth the classification of solutions of (1) is valuable in some ways.

The positive radial solutions are fairly well-understood even though the property of more general solutions is not revealed yet, see [3] for details.

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We here claim that every positive solution of (1) which decays to zero at infinity must satisfy

$$\lim_{|x|\to\infty} |x|^{2/(p-1)}u(x) = A(\frac{x}{|x|}) \tag{5}$$

for some function $A(\omega)$ on the unit sphere S^{N-1} and conversely for every nonnegative integrable function $A(\omega)$ on S^{N-1}, there exists a unique positive solution of (1) satisfying (5). This claim is not fully justified but we here announce some progress toward this main goal.

2. Behavior near infinity

We consider positive solutions of (1) satisfying

$$\lim_{|x|\to\infty} u(x) = 0. \tag{6}$$

When the spatial dimension $N = 1$, (1) becomes an ordinary differential equation. Following [2], one can show that (5) holds. With additional efforts (using the method developed in [4] and [5] for examples) one can extend this result to more general type of equations including degenerate differential equations,

$$(u^m)_{xx} + \frac{1}{\beta}xu_x + \frac{1}{p-1}u - |u|^{p-1}u = 0 \qquad \text{in} \qquad \mathbf{R}, \tag{7}$$

$$(|u_x|^{p-2})_x + \frac{1}{\gamma}xu_x + \frac{1}{q-1}u - |u|^{q-1}u = 0 \qquad \text{in} \qquad \mathbf{R}, \tag{8.}$$

Here $m \neq 1$, $p \neq 2$, $\beta = 2(p-1)/(p-m)$ and $\gamma = p(q-1)/(q-p+1)$.

Nevertheless we have not found any clues for $N \geq 2$.

3. Existence

The method which will be explained below became a fairly standard one since introduced in [3] in treating self-similar solutions.

Given a nontrivial integrable function $A(\omega)$ on S^{N-1}, we first consider a positive solution of (2) with an initial function

$$v(x,0) = A(\frac{x}{|x|})|x|^{-2/(p-1)} \qquad \text{for} \qquad x \neq 0. \tag{9}$$

The existence of such a solution is guaranteed by taking a monotone limit of a family of positive solutions $u_k(x,t)$ of (2) with truncated initial functions

$$v_k(x,0) = \min\{k, A(\frac{x}{|x|})|x|^{-2/(p-1)}\}, \tag{10}$$

which is integrable in $\mathbf{R}^N$ and its existence and uniqueness is proved in [1].

Moreover the above solution is minimal among all the positive solutions satisfying (9) and thus self-similar as shown below; Let $v(x,t)$ be the minimal solution. Then given $\lambda > 0$, obviously

$$\lambda^{2/(p-1)}v(\lambda x, \lambda^2 t) \geq v(x,t).$$

One also has for every $\mu > 0$

$$\mu^{2/(p-1)} v(\mu\lambda x, \mu^2\lambda^2 t) \geq v(\lambda x, \lambda^2 t).$$

Simply taking $\mu = 1/\lambda$, we obtain

$$v(x,t) \geq \lambda^{2/(p-1)} v(\lambda x, \lambda^2 t).$$

As remarked earlier since $v(x,t)$ is self-similar, v has the form

$$v(x,t) = t^{-1/(p-1)} w(x/\sqrt{t}), \tag{11}$$

where w satisfies (1). Moreover

$$\begin{aligned} \lim_{|x|\to\infty} |x|^{2/(p-1)} w(x) &= \lim_{|x|\to\infty} |x|^{2/(p-1)} w(\frac{x}{|x|} \cdot |x|) \\ &= \lim_{t\to 0} t^{-1/(p-1)} w(\frac{x}{|x|}/\sqrt{t}) \\ &= A(\frac{x}{|x|}). \end{aligned}$$

4. Uniqueness

When $A(\omega)$ is strictly positive for all $\omega \in S^{N-1}$, the uniqueness can be proved by applying the maximum principle as we will see below. In fact let U and u be any two solutions (we may assume that $U \geq u$).

We now define

$$l = \min\{k \geq 1 | ku(x) \geq U(x), \quad x \in \mathbf{R}^N\} \tag{12}$$

The set on the right hand side is not empty since we may take k large enough in order that the inequality holds thanks to the boundary assumption (5). The uniqueness proof is reduced to showing that l is not greater than 1.

Suppose not. From the boundary behavior at infinity, $ku(x)$ must touch $U(x)$ in a compact subset of $\mathbf{R}^N$. But $ku(x)$ is a super-solution and cannot touch $U(x)$ from above. Hence one can slightly reduce the factor l and still has the same inequality in (12), which leads to a contradiction.

We do not have any proof when $A(\omega)$ vanishes somewhere on S^{N-1}. Nevertheless the uniqueness results hold for degenerate differential equations [7] and [8] in $\mathbf{R}$, which will be announced in other space.

5. Radial solutions

In this section we remark on the nature of radial solutions of (1), that is, solutions of

$$\begin{aligned} -g'' - (\frac{r}{2} + \frac{N-1}{r})g' + |g|^{p-1}g &= \frac{1}{p-1}g, \quad r > 0 \\ g'(0) &= 0 \\ g(0) &= \alpha. \end{aligned} \tag{13}$$

We see that $g_{\alpha^*}(r) = \alpha^* = (\frac{1}{p-1})^{1/(p-1)}$ is a constant solution. When $\alpha > \alpha^*$, the corresponding solution is monotonely increasing to infinity since it is convex at every critical point. g_{α_0} is the minimal positive solution of (13) and decays at the exponential rate as $r \to \infty$. Hence it is called a fast orbit. See [2]. As well-known g_α, $\alpha_0 < \alpha < \alpha^*$, is the unique positive solution of (5.1) satisfying the decay rate

$$\lim_{r\to\infty} r^{2/(p-1)} g_\alpha(r) = C(\alpha) > 0 \tag{14}$$

and thus called a slow orbit. ¿From the uniquenes of solutions, $g_{-\alpha} = -g_\alpha$ and we are left to the case $|\alpha| < \alpha_0$, $\alpha \neq 0$. But not much is known for this case.

Let $\lambda_k = \frac{N+k-1}{2}$. The next results are the only available ones, see [3] for details.

Proposition 1. *There exist at least* $2k$ *nontrivial fast orbits for (13) if* $1/(p-1) > \lambda_{2k+1}$ *and if* $\lambda_{2k+1} < 1/(p-1) \leq \lambda_{2k+3}$ *and* $0 < \alpha < \alpha_0$*, then* $g_\alpha(r)$ *vanishes at most* $k+1$ *times for* $r > 0$*. Moreover if* g_α *vanishes exactly* $k+1$ *times for* $r > 0$*, then* g_α *becomes a slow orbit.*

In particular, Proposition 1 implies that if $N/2 < 1/(p-1) \leq 1 + N/2$ and $0 < |\alpha| < \alpha_0$, then g_α is a slow orbit since g_α changes sign for $r > 0$. Moreover we can show that g_α is in fact a slow orbit if $|\alpha|$ is relatively small.

Proposition 2. *Let* $1/(p-1) > \lambda_{2k+1}$*,* $k \geq 0$ *and* $0 < |\alpha| < \alpha^*$*. If* $|\alpha|^{p-1} \leq 1/(p-1) - \lambda_{2k+1}$*, then* g_α *vanishes at least* $k+1$ *times for* $r > 0$*.*

This result follows from the standard comparison argument, see [3] for a complete proof.

The complete understanding of the nature of solutions of (13) is very important in order to determine the large time behavior of solutions of (2) specially when the initial data changes sign because the asymptotic profile may be determined by its lap number.

References

[1] H. Brezis and A. Friedman, *Nonlinear parabolic equations involving measures as initial conditions*, J. Math. pures et appl. **62** (1983), 73-97.

[2] H. Brezis, L.A. Peletier, and D. Terman, *A very singular solution of the heat equation with absorption*, Arch. Rat. Mech. Anal. **95** (1986), 185-209.

[3] M. Kwak, *A semilinear heat equation with a singular initial data*, Proc. Roy. Soc. Edinburgh Sect. A **128** (1998), 745-758.

[4] M. Kwak, *A porous media equation with absorption. I Long time behaviour*, J. Math. Anal. Appl. **223** (1998), 96-110.

[5] M. Kwak, *A porous media equation with absorption. II Uniqueness of the very singular solution*, J. Math. Anal. Appl. **223** (1998), 111-125.

[6] Y. Naito and T. Suzuki, *Radial symmetry of self-similar solutions for semilinear heat equations*, Preprint (1999).

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