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PHASE TRANSITION LAYERED SOLUTIONS IN SINGULARLY PERTURBED SEMILINEAR ELLIPTIC NEUMANN PROBLEMS

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ABSTRACT. Existence of nonconstant solutions, their singular limits, and phase transition layered solutions for singularly perturbed semilinear elliptic Neumann problems are discussed. Some open questions or conjectures and possible future directions of research are also discussed.

1. INTRODUCTION

Many mathematicians were brought up to believe that diffusion is a smoothing and trivializing process. Indeed, this is the case for single diffusion equations. Consider the heat equation

$$(1)\qquad \begin{cases} u_t = \Delta u \text{ in } \Omega \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0 \text{ in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \times (0,\infty), \end{cases}$$

where $u_t = \partial u/\partial t$ and $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ is the usual Laplace operator, Ω is a bounded smooth domain in $\mathbb{R}^n$, ν is the unit outer normal to $\partial\Omega$, and u_0 is a real-valued continuous function(not identically zero) representing the initial heat distribution. Here the boundary condition implies that (1) is an isolated system. It is well known that the solution $u(x,t)$ of (1) becomes smooth as soon as t becomes positive and eventually converges to the constant $1/|\Omega| \int_\Omega u_0(x)dx$ as t tends to ∞. In other words, in an isolated system, no matter what the initial heat distribution is, eventually the heat distribution becomes homogeneous. A similar result holds when a source/sink term (or a reaction term) is present. That is, if we replace the linear heat equation in (1) by

$$u_t = \Delta u + f(u),$$

where f is a smooth function (linear or nonlinear), then Matano and Casten-Holland in 1978-79 proved that stable steady states must be constants provided that the domain Ω is convex. See [1]. Interesting results for nonconvex Ω have been obtained by Matano[2], Hale-Vegas, Jimbo-Morita[3,4], and others.

Let $f(u) = u(1-u^2)$. We then have an interesting asymptotic long time behavior as follows: Denoting by $S(t)$ the solution operator to the initial value problem

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with the above nonlinear equation, it follows that $S(t)$ is a continuous nonlinear semigroup and the orbit $\cup_{t>0} S(t)u_0$ is relatively compact in $W^{1,2}(\Omega)$. Thus the ω-limit set of u_0,

$$\omega(u_0) = \left\{ v \in W^{1,2}(\Omega) : \exists t_n \text{ s.t. } \lim_{t_n \to \infty} u(t_n) = v \text{ in } W^{1,2}(\Omega) \right\}$$

has the properties

(I): $\omega(u_0)$ is compact and connected;
(II): $S(t)\omega(u_0) \subset \omega(u_0)$, $\forall t \geq 0$;
(III): $\displaystyle\int_\Omega H(v)dx + \frac{1}{2}\int_\Omega |\nabla v|^2 dx$ is the same constant $\forall v \in \omega(u_0)$, where $H(v) = -\displaystyle\int_{-1}^{v} f(t)dt$;
(IV): If $v \in \omega(u_0)$, the v solves the following problem:

$$\begin{cases} \Delta u + f(u) = 0 \text{ in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ in } \partial\Omega. \end{cases}$$

It follows that if the set of solutions to the problem in (IV) is discrete, then $\omega(u_0)$ consists of just one element u^* and

$$\lim_{t \to \infty} u(t) = u^*.$$

But, if Ω is an open ball in $\mathbb{R}^n$, $n \geq 2$ and if we can prove the existence of a nonconstant solution to the problem in (IV), by the simple rotations, we may know that the set of solutions will be indiscrete. Much more information concerning the asymptotic behaviour for large t has been obtained by Nicolaenko, Scheurer and Temam[5](See also Temam[6]). In particular, they derive results about the structure of attractors and the inertial manifold. Anyway, the existence of nonconstant solutions of the problem in (IV) is a nontrivial problem and their dynamics will be chaotic.

2. Bi-stable Case

We consider the differential equation

$$u_t = \Delta u + \frac{1}{\varepsilon^2} f(u) \tag{2}$$

and assume that there exactly three zeros $z_1 < z_2 < z_3$ so that $f(z_i) = 0$ for $i = 1, 2, 3$ and $f'(z_1) < 0$, $f'(z_2) > 0$, $f'(z_3) < 0$. In this case two solutions $u = z_1$ and $u = z_3$ are stable and we call them bi-stable zeros. Formal analysis suggests the following picture: the solution $u(x, t; \varepsilon)$ separates Ω into two regions, where $u(x, t; \varepsilon) \approx z_1$ and $u(x, t; \varepsilon) \approx z_3$, respectively, and the interface between them moves with normal velocity equal to the sum of its principal curvatures. The link between (2) and motion by mean curvature was observed by Allen and Cahn[7]. As the phenomenological model, we can consider a binary alloy, comprising of species A and B, existing in a state of isothermal equilibrium at a temperature, T_m, greater than the critical temperature, T_c. The alloy's composition is spatially uniform with the concentration, u, of B taking the constant value u_m. Suppose that the alloy is now quenched(rapid reduction of temperature, that is, $\varepsilon \to 0$) to a uniform temperature T_m less than T_c. Phase separation takes place in which the composition of the alloy changes from the uniform mixed state to that of a spatially separated to phase structure, each phase being characterized by a different concentration value which is either z_1 or z_3. A much more systematic treatment was given by Rubinstein, Sternberg, and Keller[8].

The scaling of (2) has been chosen so that the associated motion by mean curvature takes place on a time scale of order one. Studying (2) is equivalent to considering the solution of

$$w_s = \varepsilon^2 \Delta w + f(w) \tag{3}$$

on a time scale of order ε^{-2}, since $w(x,s)$ solves (3) if and only if $u(x,t) = w(x, \varepsilon^{-2}t)$ solves (2).

The study of (3) is of value for understanding phase transitions. We note that this equation was proposed in [7] as a model for the motion of antiphase boundaries in crystalline solids. It is a non-conservative or "type A" Ginzburg-Landau equation, in the terminology of [9]. There are a number of other situations where Ginzburg-Landau type dynamics leads through a singular limit to a geometric model for phase boundary motion, see [10,11,12].

Viewed from the perspective of (3) we are considering an example of dynamical metastability, i.e., a pattern which persists for a long time though it eventually disappear. It is well known that as $s \to \infty$ any solution of (3) is asymptotically stationary, see [13]; for generic initial data w tends to a local minimum of the "energy"

$$J_\varepsilon(w) = \frac{\varepsilon^2}{2} \int_\Omega |\nabla w|^2 dx + \int_\Omega H(w) dx,$$

where $H(w) = -\int_{z_1}^{w} f(u)du$. The solution of (3) under consideration here are in no sense near to critical points of the energy, but they nevertheless evolve slowly, on a time scale $s \backsim \varepsilon^{-2}$. Such a phenomenon is to be expected when an evolution equation has a Liapunov function with a small parameter ε, if there are more (local) minima at $\varepsilon = 0$ than at $\varepsilon > 0$. This is the case for (3): when $\varepsilon = 0$ any measurable function w taking the value either z_1 or z_3 is a minimizer, while for $\varepsilon > 0$ the perimeter of the phase transition interface becomes important. See [14,15].

It turns out that the steady states of (3) and their limiting behaviors as $\varepsilon \to 0$ are closely related the motion by mean curvature as the singular limit of Ginzburg-Landau dynamics and that the study of the steady states of (3) essentially reduces to that of the following equation:

$$\begin{cases} \varepsilon^2 \Delta u + f(u) = 0 \text{ in } \Omega, \\ \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{4}$$

First of all, we are considering the existence of nonconstant solutions for (4). Let λ_2 be the first nonzero eigenvalue of the Laplacian Δ with the homogeneous Neumann boundary condition. If $f'(u) < \varepsilon^2 \lambda_2$ for all $z_1 \leq u \leq z_3$, then the only weak solutions u of (4) in the Sobolev space $W^{1,2}(\Omega)$ are constants $u = c$, satisfying $f(c) = 0$. See [16,17]. In the case that Ω is a dumbbell shape domain, there were only the existence results of nonconstant solutions of (4) until 1996. See [2,3].

In 1997, the author published the existence result of nonconstant solutions of (4) on general smooth bounded open domain Ω and with the sufficiently small parameter ε. See [18]. He used the well-known mountain pass theorem, see [19], and the theorem for the existence of critical points of mountain pass type assuming the existence of two minimizers, see [20], and the lemma for the existence of boundary layered solutions on Dirichlet problems, see [21,22].

Since ε is small, (4) is a singular perturbation problem. However, the traditional method, using inner and outer expansion, simply does not apply here, because the formal series expansion may not converge to the transition layered solution of (4) which traverse efficiently a boundary layer while bridging the values near z_1 and z_3 exponentially. To solve (4) we need only to find nontrivial critical points of J_ε. This energy consideration of solutions turns out to be the new key ingredient. Since this functional J_ε has already two global minimizers $w = z_1$ and $w = z_3$ and the critical point $w = z_2$, we search for a nonconstant saddle point of J_ε via a variational approach. There is an excellent reference about variational methods. See [23].

Theorem 2.1 ([18]). *Assume the conditions:*

(f1): $f : [z_1, z_2] \to \mathbb{R}^1$ *is of class* C^1*;*

(f2): *there are exactly three zeros,* $z_1 < z_2 < z_3$ *so that* $f(z_i) = 0$*, for* $i = 1, 2, 3$*, and* $f'(z_1) < 0, f'(z_2) > 0, f'(z_3) < 0$*;*

(f3): $\displaystyle\int_{z_2}^{z_1} f(u)du = \int_{z_2}^{z_3} f(u)du.$

Then there is a small number $\varepsilon_0 > 0$ *so that, for any* $0 < \varepsilon < \varepsilon_0$*, there is a nonconstant solution* $u_1(x;\varepsilon)$ *to the problem (4) so that it converges to the value either* z_1 *or* z_3 *almost everywhere in* Ω *as* $\varepsilon \to 0$*.*

We first have an interesting mathematical problem. What is the singular limit of $u_1(x;\varepsilon)$ as $\varepsilon \to \infty$? What kinds of layers in u_1 can we expect? For examples, phase transition layers, or spike layers, boundary layers, nonuniform interior transition layers, etc? As we know, these kinds of limiting behaviors are not fully understood at this time. In 1989, Kohn and Sternberg[25] constructed local minimizers to J_ε, for all sufficiently small ε, near each isolated local minimizers of the associated geometry problem. These solutions have phase transition layers as $\varepsilon \to \infty$. In the author's opinion, there are no any other similar results to Theorem 2.2 so far. Their methods were based on the relationship between J_ε and a geometry problem F_0: The minimization of the following perimeter functional

$$F_0(u) = \begin{cases} \mathrm{Per}_\Omega \{x \in \Omega : u(x) = z_3\} \text{ if either } u(x) = z_1 \text{ or } u(x) = z_3, \\ \\ \infty \text{ otherwise,} \end{cases}$$

where Per_Ω is defined by

$$\mathrm{Per}_\Omega(A) = \sup\left\{\int_A \mathrm{div}(g) : g = (g_1, g_2, \cdots, g_n) \in C_0^\infty(\Omega), \sum g_i^2 \le 1\right\}.$$

To study the relationship, they use the theory of Γ-convergence which was invented by DeGiorgi.

Theorem 2.2 ([25]). *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *with Lipschitz boundary, and suppose that* u_0 *is an isolated* L^1*-local minimizer of* F_0*. Then there exists* $\varepsilon_0 > 0$ *and a family* $\{u_\varepsilon\}_{\varepsilon<\varepsilon_0}$ *such that* u_ε *is an* L^1*-local minimizer of* J_ε *and* $||u_\varepsilon - u_0||_{L^1(\Omega)} \to 0$ *as* $\varepsilon \to 0$*.*

The above theorem heavily depends on the geometry of Ω which is a very special dumbbell shape. The isolation of local minimizers is also crucial. We may have the following conjecture: Given any minimizer u_0 of F_0, is there a sequence of critical points u_ε of J_ε so that $||u_\varepsilon - u_0||_{L^1(\Omega)} \to 0$ as $\varepsilon \to 0$? This problem is widely open.

The above phase transition layered solutions may be used to calculated free energy of the Gibbs-Thompson relation(Thermodynamics) within the gradient theory of phase transitions in a microscopic theory for antiphase boundary motions. See [7]. From this point of view, we have the following open question: Given any measurable function u_0 which has the value z_1 or z_3 only and may not be a local minimizer of F_0, is there a sequence of critical points u_ε of J_ε so that $||u_\varepsilon - u_0||_{L^1(\Omega)} \to 0$ as $\varepsilon \to 0$?

If we replace the equality in the condition (f3) by the inequality, using Taheri's result[24] for the local minimizers $w = z_1$ and $w = z_3$ of J_ε, we have the following theorem:

Theorem 2.3 ([18]). *Assume* (f1) *and* (f2)*. Suppose that* $\int_{z_2}^{z_1} f(u)du < \int_{z_2}^{z_3} f(u)du$*. Then for any sufficiently small* $\varepsilon > 0$ *there exists a nonconstant solution* $u_2(x;\varepsilon)$ *of (4) so that it converges to* z_1 *almost everywhere in* Ω *as* $\varepsilon \to 0$*.*

We then also have an interesting mathematical problem: What is the singular limit of $u_2(x;\varepsilon)$ as $\varepsilon \to 0$? We may expect several kinds of spike layered solutions from the limiting behavior in Theorem 2.3. However, this problem is also widely open. In fact, recent research [26] shows that spike layers appears in Cahn-Hilliard equation which is similar to (4). Although spike layered solutions there perhaps are unstable, they seem to have profound implications on the dynamics involved.

3. Bi-unstable Case

We replace the condition $(f2)$ on f in Theorem 2.1 and Theorem 2.3 by the followings: $z_1 < z_2 < z_3, f(z_i) = 0$ for $i = 1, 2, 3$, and $f'(z_1) > 0, f'(z_2) < 0, f'(z_3) > 0$. This means that z_1 and z_3 are unstable. In particular, let $f(u) = -u + u|u|^{p-1}$ with $p > 1$. In this case, there are very interesting mathematical results for spike layered positive solutions of (4). Ni published an excellent survey paper about that. See [27].

4. Concluding Remark

Mathematically, it seems interesting to replace the homogeneous Dirichlet boundary condition $u = 0$ on $\partial\Omega$ in the problem (4). Furthermore, since the Dirichlet boundary condition is far more rigid than the Neumann boundary condition, it follows much fewer solutions to exist. It would be very interesting to understand how solutions change while the Dirichlet boundary condition $u = 0$ is continuously deformed to the Neumann condition $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, say, via

$$(1-r)u + r\frac{\partial u}{\partial \nu} = 0$$

on $\partial\Omega$, where r varies from 0 to 1.

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