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STABILITY OF ANALYTIC OPERATOR-VALUED FUNCTION SPACE INTEGRALS

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Abstract. Since the Feynman integral was introduced by Feynman in 1948, there has been considerable progress on stability of Feynman integrals in recent years. The first stability theorem for the integral was introduced by Johnson in 1984 as a bounded linear operator on $L_2(\mathbb{R})$ and it was extended to the case of $L_p(\mathbb{R})$ with respect to measures, potentials and wave of functions. In this paper, we will briefly review the previous results and introduce the recent results on the stability of analytic operator-valued function space integrals.

1. Introduction

In 1984, Johnson proved a bounded convergence theorem for the operator-valued function space integral [7]. This is the first stability theorem for the integral as a bounded linear operator on $L_2(\mathbb{R}^n)$ where n is any positive integer. In [10], Johnson and Skoug introduced stability theorems for the integral as an $\mathcal{L}(L_p(\mathbb{R}^N), L_{p'}(\mathbb{R}^N))$ theory, $1 < p \leq 2$. Chang studied stability theorems for the integral as a bounded linear operator from $L_1(\mathbb{R})$ to $C_0(\mathbb{R})$ [2]. In those papers mentioned above, they treated certain functionals which involve only the Lebesgue measure on the interval $(0, t)$.

In [8], Johnson and Lapidus established stability theorems for the integral as an $\mathcal{L}(L_2(\mathbb{R}^N), L_2(\mathbb{R}^N))$ theory for certain functionals involving any Borel measures on $(0, t)$. Chang and Ryu proved theorems insuring stability with respect to potentials and wave functions for the integral as a bounded linear operator on $L_p(\mathbb{R}^N)$ for certain functionals involving some Borel measures on $(0, t)$ [5].

Lapidus provided a nice stability theorem for the modified Feynman integral [11]. The modified Feynman integral is a variation of Nelson's approach to the Feynman integral via the Trotter product formula [12].

Recently, Chang, Ko, and Ryu studied the stability of the operator-valued function space integral as an $\mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$ theory for the functionals involving some Borel measures on $(0, t)$ with respect to potentials, wave functions and measures [4].

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2. Stability Theorems involving the Lebesgue Measure

For our present purposes, $t > 0$ will be fixed. $C[0,t]$ will denote the space of $\mathbb{R}$-valued continuous functions on $[0,t]$. $C_0[0,t]$ denotes the space of functions x in $C[0,t]$ such that $x(0) = 0$. $C^\nu[0,t]$ will denote the product of ν copies of $C[0,t]$ and $C_0^\nu[0,t]$, respectively. m will be referred to as Wiener measure and m_ν will denote the product of ν copies of 1-dimensional Wiener measure.

Let $1 < p \le 2$ be given and let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Given $\lambda > 0$, ψ in $L_p(\mathbb{R}^\nu)$, and ξ in $\mathbb{R}^\nu$, let

$$(I_\lambda(F)\psi)(\xi) = \int_{C_0^\nu[0,t]} F(\lambda^{-\frac{1}{2}}x + \xi)\psi(\lambda^{-\frac{1}{2}}x(t) + \xi)\ dm_\nu(x).$$

This formula may define, for each $\lambda > 0$, a bounded linear operator $I_\lambda(F)$ in $\mathcal{L} = \mathcal{L}(L_p(\mathbb{R}^\nu), L_{p'}(\mathbb{R}^\nu))$, the space of bounded linear operator from $L_p(\mathbb{R}^\nu)$ to $L_{p'}(\mathbb{R}^\nu)$. If this is so, and if the operator-valued function $\lambda \to I_\lambda(F)$ has an analytic continuation to $\mathbb{C}^+ := \{\lambda \in \mathbb{C} : Re\lambda > 0\}$, we denote this analytic continuation $I_\lambda^{an}(F)$. Finally given a real parameter q, $q \neq 0$, the operator-valued, analytic Feynman integral $J_q^{an}(F)$ is defined by

$$J(F) = J_q^{an}(F) := \lim_{\lambda \to -iq} I_\lambda^{an}(F)$$

where the limits is taken in the strong operator topology and where λ approaches $-iq$ through $\mathbb{C}^+$.

Let θ be a bounded complex-valued Lebesgue measurable function on $\mathbb{R}^\nu$, and let

$$F(x) = \exp\left\{\int_0^t \theta(x(s))\ ds\right\}. \tag{2.1}$$

Theorem 2.1. *Let θ be an essentially bounded, complex-valued, Lebesgue measurable function on $\mathbb{R}^\nu$ and let F be given by* (2.1). *Then $J(F)$ exists as a bounded linear operator on $L_2(\mathbb{R}^\nu)$.*

The following theorem is the first stability theorem for the Feynman integral introduced by Johnson in 1984 [7].

Theorem 2.2 (Stability Theorem for L_2 case). *Let $\{\theta_m\}$ be a sequence of complex-valued, Lebesgue measurable functions on $\mathbb{R}^\nu$ all of which are essentially bounded by the number L. Suppose that $\theta_m \to \theta$ a.e. on $\mathbb{R}^\nu$. Then θ is also essentially bounded by L and, by Theorem 2.1, $J(F)$ and $J(F_m)$, $m = 1, 2, \cdots$, all exist, where F is given by* (2.1) *and F_m is given by replacing θ by θ_m in* (2.1). *Further, $J(F_m)$ converges to $J(F)$ in the strong operator topology.*

Now we introduce the stability result for L_p case. Let the numbers p, ν, γ and r satisfy the following restrictions :

(a) Let $1 < p \le 2$ be given and let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$.

(b) $\gamma = \frac{p}{2-p}$. Note that $\gamma = \gamma(p)$ is an increasing function taking the interval $(1, 2]$ onto the interval $(1, +\infty]$. We remark that γ is chosen so that multiplication by a function in $L_\gamma(\mathbb{R}^\nu)$ is a bounded linear operator from $L_{p'}(\mathbb{R}^\nu)$ to $L_p(\mathbb{R}^\nu)$.

(c) ν is a positive integer satisfying $1 \le \nu < 2\gamma$.

(d) $\frac{2\gamma}{2\gamma-\nu} < r \le +\infty$ except that we allow $1 \le r \le +\infty$ when $p = 2$ and $\gamma = +\infty$.

The mixed norm space $L_{\gamma r} = L_{\gamma r}([0,t]\times\mathbb{R}^\nu)$ will play a key role throughout. A complex-valued Lebesgue measurable function g on $[0,t]\times\mathbb{R}^\nu$ is said to be in $L_{\gamma r}$ if and only if $g(s,\cdot)$ is in $L_\gamma(\mathbb{R}^\nu)$ for almost every (a.e.) s in $[0,t]$ and $\|g(s,\cdot)\|_\gamma$ is in $L_r([0,t])$. Note that $L_{\gamma r_2}\subseteq L_{\gamma r_1}$ if $1\le r_1\le r_2\le +\infty$.

Given θ and θ_m, $m=1,2,\cdots$ from $L_{\gamma r}([0,t]\times\mathbb{R}^\nu)$, let F and F_m , $m=1,2,\cdots$ be defined on $C^\nu[0,t]$ by

$$F(x)=\exp\left\{\int_0^t \theta(t-s,x(s))\right\}\,ds \tag{2.2}$$

$$F_m(x)=\exp\left\{\int_0^t \theta_m(t-s,x(s))\right\}\,ds, \tag{2.3}$$

where θ is the potential function which is of primary interested in quantum mechanics.

Theorem 2.3 (Stability theorem for L_p case). *Let h be in $L_{\gamma r}$. Let $\{\theta_m\}$ be a sequence from $L_{\gamma r}$ such that $\theta_m(s,u)\to\theta(s,u)$ for a.e. (s,u) and such that, for every m, $|\theta_m(s,u)|\le h(s,u)$ for a.e. (s,u). Then θ is also in $L_{\gamma r}$ and $J(F)$ and $J(F_m)$, $m=1,2,\cdots$ all exist as elements of $\mathcal{L}(L_p(\mathbb{R}^\nu),L_{p'}(\mathbb{R}^\nu))$. Further, $J(F_m)$ converges to $J(F)$ in the strong operator topology.*

Remark. Theorem 2.2 is the special case of Theorem 2.3 where $p=p'=2$, $\gamma=+\infty$, and θ,θ_m's are time-independent.

3. Stability Theorems involving Borel measures

Let $\mathbb{C}_+$ and $\mathbb{C}_+^\sim$ be the set of all complex numbers with positive real part and all nonzero complex numbers with nonnegative real part, respectively.

Let $t>0$ be given. $M(0,t)$ will denote the space of complex Borel measures on the interval $(0,t)$. A measure μ in $M(0,t)$ is said to be continuous if $\mu(\{\tau\})=0$ for every τ in $(0,t)$ and a measure ν in $M(0,t)$ is said to be discrete if there is an at most countable subset $\{\tau_i : i=1,2,\cdots\}$ of $(0,t)$ and a summable sequence $\langle\omega_i\rangle$ from $\mathbb{C}$ such that $\nu=\sum_{i=1}^\infty \omega_i\delta_{\tau_i}$ where δ_{τ_i} is the Dirac measure with total mass one concentrated at τ_i. Then every measure η in $M(0,t)$ has a unique decomposition, $\eta=\mu+\nu$ into a continuous part μ and a discrete part ν.

$M(0,t)^*$ will denote the subset of $M(0,t)$ which satisfies the following conditions:
(a) If μ is the continuous part of η in $M(0,t)$, then the Radon-Nikodym derivative $d|\mu|/dm$ exists and is essentially bounded where m is the Lebesgue measure on $(0,t)$.
(b) If $\nu=\sum_{i=1}^\infty\omega_i\delta_{\tau_i}$ is the discrete part of η in $M(0,t)$, then $\sum_{i=1}^\infty|\omega_i|\tau_i^{-\gamma'\delta}$ converges where $\delta=N/2\alpha$, $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$.

For η in $M(0,t)^*$, let $L_{\alpha\gamma:\eta}([0,t]\times\mathbb{R}^N)\equiv L_{\alpha\gamma:\eta}$ be the space of $\mathbb{C}$-valued Borel measurable functionals θ on $[0,t]\times\mathbb{R}^N$ such that

$$\|\theta\|_{\alpha\gamma:\eta}\equiv\left\{\int_{(0,t)}\|\theta(s,\cdot)\|_\alpha^\gamma\,d|\eta|(s)\right\}^{1/\gamma}<\infty.$$

Let F be a functional on $C[0,t]$. Given $\lambda>0$, ψ in $L_p(\mathbb{R}^N)$, and ξ in $\mathbb{R}^N$, let

$$[I_\lambda(F)\psi](\xi)=\int_{C_0[0,t]}F(\lambda^{-1/2}x+\xi)\psi(\lambda^{-1/2}x(t)+\xi)\,dm_w(x),$$

where $C[0,t]$ is the space of $\mathbb{R}^N$-valued continuous functions on $[0,t]$, $C_0[0,t] = \{x \in [0,t] : x(0) = 0\}$ and m_w is the Wiener measure. If for m_L-a.e. ξ in $\mathbb{R}^N$, $[I_\lambda(F)\psi](\xi)$ exists in $L_{p'}(\mathbb{R}^N)$ and if the map $\psi \to I_\lambda(F)\psi$ gives an element of $\mathcal{L}(L_p(\mathbb{R}^N), L_{p'}(\mathbb{R}^N))$, we say that the operator-valued function space integral $I_\lambda(F)$ exists for λ. Suppose there exists λ_0 $(0 < \lambda_0 \leq \infty)$ such that $I_\lambda(F)$ exists for all $0 < \lambda < \lambda_0$ and there exists an $\mathcal{L}(L_p(\mathbb{R}^N), L_{p'}(\mathbb{R}^N))$-valued function which is analytic in $\mathbb{C}_{+,\lambda_0} \equiv \mathbb{C}_+ \cap \{z \in \mathbb{C} : |z| < \lambda_0\}$ and agrees with $I_\lambda(F)$ on $(0, \lambda_0)$, then this $\mathcal{L}(L_p, L_{p'})$-valued function is called the operator-valued function space integral of F associated with λ and in this case, we say that $I_\lambda(F)$ exists for λ in $\mathbb{C}_{+,\lambda_0}$. If $I_\lambda(F)$ exists for λ in $\mathbb{C}_{+,\lambda_0}$ and $I_\lambda(F)$ is strongly continuous in $\mathbb{C}^{\sim}_{+,\lambda_0} \equiv \mathbb{C}^{\sim}_+ \cap \{z \in \mathbb{C} : |z| < \lambda_0\}$, we say that $I_\lambda(F)$ exists for λ in $\mathbb{C}^{\sim}_{+,\lambda_0}$. When λ is purely imaginary, $I_\lambda(F)$ is called the analytic operator-valued Feynman integral of F.

First, we consider stability with respect to the potentials.

Let η be in $M(0,t)^*$ and let H belong to $L_{\alpha\gamma:\eta}$. Let $\eta = \mu + \sum_{i=1}^{\infty} \omega_i \delta_{\tau_i}$ be the decomposition of η into its continuous part and discrete part and let $\langle \theta_n \rangle$ be a sequence in $L_{\alpha\gamma:\eta}$. For nonnegative integer n, we let

$$F_n(y) = \Big(\int_{(0,t)} \theta(s, y(s))\, d\eta(s) \Big)^n \quad \text{for } y \text{ in } C[0,t] \tag{3.1}$$

and

$$F_n^{(m)}(y) = \Big(\int_{(0,t)} \theta_m(s, y(s))\, d\eta(s) \Big)^n \quad \text{for } y \text{ in } C[0,t]. \tag{3.2}$$

Theorem 3.1. *Suppose $\theta_m(s,u) \to \theta(s,u)$ as $m \to \infty$ for $\eta \times m_L$-a.e. (s,u) and $|\theta_m(s,u)| \leq H(s,u)$ for all positive integer m and for $\eta \times m_L$-a.e. (s,u). Then θ is also in $L_{\alpha\gamma:\eta}$ and for all positive integers m, n, $I_\lambda(F_n)$ and $I_\lambda(F_n^{(m)})$ exist for all λ in $\mathbb{C}^{\sim}_+$ and*

$$I_\lambda(F_n^{(m)}) \to I_\lambda(F_n) \quad \textit{strongly as } m \to \infty.$$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in $\mathbb{C}^{\sim}_+ \cap \{z \in \mathbb{C} : |z| < \lambda_0\}$ for some positive real number λ_0. For positive integer m and y in $C[0,t]$, let

$$F(y) = f\Big(\int_{(0,t)} \theta(s, y(s))\, d\eta(s) \Big) \quad \text{and} \tag{3.3}$$

$$F^{(m)}(y) = f\Big(\int_{(0,t)} \theta_m(s, y(s))\, d\eta(s) \Big). \tag{3.4}$$

Theorem 3.2. *Suppose for all λ in $\mathbb{C}^{\sim}_{+,\lambda_0}$ and for all positive integer m, $\sum_{n=1}^{\infty} |a_n| \| I_\lambda(F_n^{(m)}) \|$ and $\sum_{n=1}^{\infty} \| I_\lambda(F_n) \|$ are convergent. Then $I_\lambda(F)$ and $I_\lambda(F^{(m)}), m = 1, 2, \cdots,$ exists for all λ in $\mathbb{C}^{\sim}_{+,\lambda_0}$ and*

$$I_\lambda(F^{(m)}) \to I_\lambda(F) \quad \textit{strongly as } m \to \infty \textit{ in } \mathbb{C}^{\sim}_{+,\lambda_0}.$$

Now, we consider stability with respect to the wave functions.

Theorem 3.3. *Suppose $\psi^{(m)} \to \psi$ in $L_p(\mathbb{R}^N)$ as $m \to \infty$. Then for positive integer m, $I_\lambda(F)\psi$ and $I_\lambda(F^{(m)})\psi^{(m)}$ exist in $L_{p'}(\mathbb{R}^N)$ for λ in $\mathbb{C}^{\sim}_{+,\lambda_0}$ and $I_\lambda(F^{(m)})\psi^{(m)} \to I_\lambda(F)\psi$ in $L_{p'}(\mathbb{R}^N)$ as $m \to \infty$.*

Lastly, we treat the stability theorem with respect to the measures. Let η and $\eta_m, m = 1, 2, \cdots$, be in $M(0,t)^*$ such that η_m converges to η in the total variation norm and let θ be bounded which is in $L_{\alpha\gamma:\eta} \cap (\cap_{m=1}^{\infty} L_{\alpha\gamma:\eta_m})$. Let

$$F_n(y) = \Big(\int_{(0,t)} \theta(s, y(s))\, d\eta(s)\Big)^n \quad \text{and} \tag{3.5}$$

$$F_n^{(m)}(y) = \Big(\int_{(0,t)} \theta(s, y(s))\, d\eta_m(s)\Big)^n \quad \text{for } y \text{ in } C[0,t]. \tag{3.6}$$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and let

$$F^m(y) = f\Big(\int_{(0,t)} \theta(s, y(s))\, d\eta_m(s)\Big) \quad \text{for } y \text{ in } C[0,t]. \tag{3.7}$$

Theorem 3.4. *Suppose that there are positive real number λ_0 and K such that for all λ in $\mathbb{C}^{\sim}_{+,\lambda_0}$ and all positive real number m,*

$$\sum_{n=0}^{\infty} |a_n| \|I_\lambda(F_n^m)\| < K \ \textit{and} \ \sum_{n=0}^{\infty} |a_n| \|I_\lambda(F_n)\| < K.$$

Then for λ in $\mathbb{C}^{\sim}_{+,\lambda_0}$ and for positive integer m, $I_\lambda(F)$ and $I_\lambda(F^m)$ exist and $I_\lambda(F^m) \to I_\lambda(F)$ uniformly in λ on all compact subsets of $\mathbb{C}_{+,\lambda_0}$ in the operator norm topology. Moreover for all $0 < \lambda < \lambda_0$,

$$\|I_\lambda(F^m) - I_\lambda(F)\| \le \frac{\lambda_0}{2\pi t} T_m \quad \textit{for } m = 1, 2, \cdots$$

where $T_m = \sup\{|f(z_1) - f(z_2)| : |z_1 - z_2| \le M_m\}$ with $M_m = \|\eta - \eta_m\| \|\theta\|_\infty$. Here, $\|\theta\|_\infty$ denotes the supremum norm of θ on $(0,t) \times \mathbb{R}^N$.

4. Stability Theorems: $\mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$ Theory

Let $C_0(\mathbb{R})$ will denote the space of $\mathbb{C}$-valued continuous functions on $\mathbb{R}$ which vanish at ∞ with the supremum norm. $L_1(\mathbb{R})$ is the space of Borel measurable, $\mathbb{C}$-valued functions ψ on $\mathbb{R}$ such that $|\psi|$ is integrable with respect to the Lebesgue measure m on $\mathbb{R}$ with the norm $\|\psi\|_1 = \int |\psi|\, dm$. $\mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$ will denote the space of bounded linear operators from $L_1(\mathbb{R})$ to $C_0(\mathbb{R})$. Let $\tilde{M}(0,t)$ denote the space of complex Borel measures η on the interval $(0,t)$ which satisfy the following conditions;

(1) If μ is the continuous part of η, the Radon-Nikodym derivative $\frac{d|\mu|}{dm}$ exists and is essentially bounded, where m is the Lebesgue measure on $(0,t)$.

(2) $\eta = \sum_{j=1}^{k} w_j \delta_{\tau_j} + \mu$, where δ_{τ_j} is the Dirac measure at $\tau_j \in (0,t)$, $0 < \tau_1 < \cdots < \tau_k < t$ and $w_j \in \mathbb{C}$ for $j = 1, 2, \cdots, k$.

Let $r \in (2,\infty]$ and $\eta \in \tilde{M}(0,t)$. Let $L_{1r:\eta}([0,t]\times\mathbb{R}) \equiv L_{1r:\eta}$ be the space of all Borel measurable $\mathbb{C}$-valued functions θ on $[0,t]\times\mathbb{R}$ such that

$$\| \theta \|_{1r:\eta} \equiv \left\{ \int_{(0,t)} \| \theta(s,\cdot) \|_1^r \, d|\eta|(s) \right\}^{1/r}$$

is finite. If θ is in $L_{1r:\eta}$ and $\eta = \mu + \nu$ is the Lebesgue decomposition, it is not difficult to show that $\theta \in L_{1r:\mu} \cap L_{1r:\nu}$. Let $\eta \in \tilde{M}(0,t)$. A Borel measurable $\mathbb{C}$-valued function θ on $[0,t]\times\mathbb{R}$ is said to belong to $L_{\infty 1:\eta}$ if

$$\|\theta\|_{\infty 1:\eta} = \int_{(0,t)} \|\theta(s,\cdot)\|_\infty \, d|\eta|(s)$$

is finite.

Let F be a functional from $C[0,t]$ to $\mathbb{C}$. Given $\lambda > 0, \psi \in L_1(\mathbb{R})$ and $\xi \in \mathbb{R}$, let

$$(I_\lambda(F)\psi)(\xi) = \int_{C_0[0,t]} F(\lambda^{-1/2}x + \xi)\psi(\lambda^{-1/2}x(t) + \xi)\, dm(x).$$

If $I_\lambda(F)\psi$ is in $C_0(\mathbb{R})$ as a function of ξ and if the correspondence $\psi \to I_\lambda(F)\psi$ gives an element of $\mathcal{L} \equiv \mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$, we say that the operator-valued function space integral $I_\lambda(F)$ exists. Next suppose that there exists λ_0 $(0 < \lambda_0 < \infty)$ such that $I_\lambda(F)$ exists for all λ in $(0,\lambda_0)$ and further suppose that there exists an $\mathcal{L}$-valued function which is analytic in $\mathbb{C}_{+,\lambda_0}$ and agree with $I_\lambda(F)$ on $(0,\lambda_0)$. Then this $\mathcal{L}$-valued function is denoted by $I_\lambda^{an}(F)$ and is called the operator-valued analytic Wiener integral of F associated with λ. Finally, let q be in $\mathbb{R}$ with $0 < |q| < \lambda_0$. Suppose there exists an operator $J_q^{an}(F)$ in $\mathcal{L}$ such that for every ψ in $L_1(\mathbb{R})$, $J_q^{an}(F)\psi$ is the weak limit of $I_\lambda^{an}(F)\psi$ as $\lambda \to -iq$ through $\mathbb{C}_{+,\lambda_0}$. Then $J_q^{an}(F)$ is called the operator-valued function space integral of F associated with q.

Firstly, we establish the stability for the operator-valued Feynman integral of functionals involving some Borel measures on $(0,t)$ with respect to potentials.

Theorem 4.1. *Let η be in $\tilde{M}(0,t)$ with $\eta = \mu + \sum_{p=1}^{k} w_p\delta_{\tau_p}$. Let $H \in L_{1r:\eta}$ and $H(\tau_p,\cdot)$ be essentially bounded for each $p = 1,2,\cdots,k$. Let $\theta^{(N)}, N = 1,2,\cdots,$ be Borel measurable functions on $[0,t]\times\mathbb{R}$ such that for $\eta\times m$-a.e.*

$$\theta^{(N)} \longrightarrow \theta \quad \text{as } N \to \infty$$

and

$$|\theta^{(N)}| \le |H| \quad \text{for } N = 1,2,\cdots.$$

Then θ and $\theta^{(N)}$ belong to $L_{1r:\eta}$. For nonnegative integer n, let

$$F_n(x) = \Big(\int_{(0,t)} \theta(s,x(s))\, d\eta(s)\Big)^n \quad \text{for } x \text{ in } C[0,t]$$

and

$$F_n^{(N)}(x) = \Big(\int_{(0,t)} \theta^{(N)}(s,x(s))\, d\eta(s)\Big)^n \quad \text{for } x \text{ in } C[0,t].$$

Then for all real $q > 0$, $J_q^{an}(F_n)$ *and* $J_q^{an}(F_n^{(N)})$ *exist for each* $N \in \mathbb{N}$ *and as* $N \to \infty$,

$$J_q^{an}(F_n^{(N)}) \to J_q^{an}(F_n) \quad \textit{in the operator norm.}$$

Let

$$F(y) = f\left(\int_{(0,t)} \theta(s, y(s))\, d\eta(s)\right) \quad \text{for } y \text{ in } C[0,t]$$

and

$$F^{(N)}(y) = f\left(\int_{(0,t)} \theta^{(N)}(s, y(s))\, d\eta(s)\right) \quad \text{for } y \text{ in } C[0,t].$$

Now, we consider the stability for the operator-valued Feynman integral of functionals with respect to wave functions.

Theorem 4.2. *Let* $\{\psi^{(N)}\}$ *be a sequence in* $L_1(\mathbb{R})$ *and* $\| \psi^{(N)} - \psi \|_1 \to 0$ *as* $N \to \infty$. *Then for* $N \in \mathbb{N}$, $J_q^{an}(F)\psi$ *and* $J_q^{an}(F^{(N)})\psi^{(N)}$ *exist in* $C_0(\mathbb{R})$ *for real* q *with* $0 < |q| < \lambda_0$. *Moreover,*

$$\|J_q^{an}(F^{(N)})\psi^{(N)} - J_q^{an}(F)\psi\|_\infty \to 0 \quad \textit{as } N \to \infty.$$

Finally, we treat the stability theorem for the operator-valued Feynman integral with respect to measures.

Theorem 4.3. *Let* θ *be a continuous function bounded by c and let* η *and* η_N, $N = 1, 2, \cdots$ *be in* $\tilde{M}(0,t)$. *Assume that*

$$\eta_N \to \eta \quad \textit{weakly.}$$

Let

$$F_N(y) = f\left(\int_{(0,t)} \theta(s, y(s))\, d\eta_N(s)\right) \quad \textit{for } y \textit{ in } C[0,t].$$

Then

$$I_\lambda^{an}(F_N) \to I_\lambda^{an}(F) \quad \textit{in the operator norm,}$$

uniformly in λ *on all compact subset of* $\mathbb{C}_{+,\lambda_0}$.

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