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OSCILLATION THEORY FOR DELAY AND NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we will survey the recent results about oscillation for the first order delay differential equation

$$x'(t) + p(t)x(t-\tau) = 0,\ t \geq t_0$$

and the first order neutral delay differential equation

$$\frac{d}{dt}[x(t) + p(t)x(t-\tau)] + q(t)x(t-\sigma) = 0,\ t \geq t_0.$$

1. Introduction

Since Sturm (1836) posed oscillation problems of the second order linear differential equation

$$x''(t) + a(t)x(t) = 0$$

when he investigated thermal conductivity, the oscillation theory for differential equations has undergone a very long history. Hutchinson (1948) established the delay logistic equation for single species :

$$N'(t) = aN(t)[1 - \frac{N(t-\tau)}{K}], \qquad \text{(E - 1)}$$

where the delay quantity τ includes various factors influencing the increase of species such as the hatching period, pregnant time and the time of renewal of food. Based on biological considerations ecologists predict that there are solutions with a small positive initial value for (E-1), which will steadily approach to $N(t) = K$, when $0 < a,\ \tau \ll 1$, but for a large τ, the solution may exceed K and start oscillating around K. This oscillatory phenomenon had already been observed in the lab. (E-1) becomes

$$x'(t) + c[1 + x(t)]x(t-1) = 0. \qquad \text{(E - 2)}$$

by using variable transformation

$$x(t) = \frac{N(\tau t)}{K} - 1,\ c = a\tau$$

in (E-1). It is known that if $c = a\tau > \frac{1}{e}$, then every solution of (E-2) is oscillatory, meaning that for an arbitrary large T the solution $x(t)$ changes sign at $t \geq T$. This

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mathematical analysis affords ecologistic limits for the delay quantity τ, which causes oscillatory phenomenon.

A solution $x(t)$, $t \geq t_0$, of a differential equation is called *oscillatory* if there exists a sequence $(t_n) \to \infty$ as $n \to \infty$ such that $x(t_n) = 0$, i.e., it has arbitrarily large zeros, and it is called *nonoscillatory* if there exists a $T \geq t_0$ such that $|x(t)| > 0$ for $t > T$. A differential equation is said to be *oscillatory* if all the solutions are oscillatory.

As a simple example, the equation

$$x'(t) + x(t) = 0$$

has a nonoscillatory solution $x(t) = e^{-t}$ but the delay equation

$$x'(t) + x(t - \frac{\pi}{2}) = 0$$

has an oscillatory solution $x(t) = \sin t$. This oscillation is completely caused by the delay of $\tau = \frac{\pi}{2}$. Of course, the nonoscillation is also caused by the delay.

The main topics in oscillation theory of differential equations are the following.

(i) Finding conditions for all solutions to be oscillatory.

(ii) Finding conditions for the existence of nonoscillatory solutions.

(iii) Studying the classification of nonoscillatory solutions and the existence criteria of various nonoscillatory solutions.

(iv) Finding linearized oscillation criteria.

(v) Studying the behavior of oscillatory solutions.

(vi) Studying the stability theory related to the oscillation theory.

In this paper we will survey the recent results about oscillation for the first order delay differential equation

$$x'(t) + p(t)x(t - \tau) = 0, \quad t \geq t_0$$

and the first order neutral delay differential equation

$$\frac{d}{dt}[x(t) + p(t)x(t - \tau)] + q(t)x(t - \sigma) = 0, \ t \geq t_0.$$

2. Delay Differential Equations

Consider the delay equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{E - 3}$$

where $p \in C([t_0, \infty), \mathbb{R}^+)$ and $\tau(t) < t$. The first systematic investigation for oscillation of (E-3) was studied by Myskis (1950). He obtained that (E-3) is oscillatory if

$$\lim_{t\to\infty} \sup[t - \tau(t)] < \infty, \tag{2 - 1}$$

$$\lim_{t\to\infty} \inf[t - \tau(t)] \cdot \lim_{t\to\infty} \inf p(t) > \frac{1}{e}.$$

The well-known improvement of the conditions (2-2) was posed by Ladas in 1979 :

$$L = \lim_{t\to\infty} \inf \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}. \qquad (2 - 2)$$

For example,

$$x'(t) + (\sin t)x(t - 2\pi) = 0, \; t \geq 0$$

is nonoscillatory and a nonoscillatory solution is $x(t) = e^{\cos t}$.

In 1995, Li [10] improved the condition (2-2) as follows :

$$\int_{t-\tau}^{t} p(s)ds > \frac{1}{e}, \quad t \geq t_0 + \tau, \qquad (2 - 3)$$

$$\int_{t_0+\tau}^{\infty} p(t)(\int_{t-\tau}^{t} p(s)ds - \frac{1}{e})dt = \infty.$$

He showed that the delay equation

$$x'(t) + p(t)x(t - \tau) = 0, \quad t \geq t_0, \; p(t) \geq 0 \qquad (E - 4)$$

is oscillatory under the conditions (2-3). Furthermore, Li [11] replaced the conditions (2-3) by

$$\int_{t}^{t+\tau} p(s)ds > 0, \; t \geq t_0 > 0, \qquad (2 - 4)$$

$$\int_{t_0}^{\infty} p(t)\log(e\int_{t}^{t+\tau} p(s)ds)dt = \infty.$$

Now, we consider the following delay equation

$$x'(t) + p(t)x(t - \tau(t)) = 0, \quad t \geq 0, \qquad (E - 5)$$

where $p, \tau \in C(\mathbb{R}^+)$. We choose a function

$$T(t) = t - \tau(t), \quad t \geq 0$$

with $\lim_{t\to\infty} T(t) = \infty$ and denote

$$L = \lim_{t\to\infty} \inf \int_{T(t)}^{t} p(s)ds, \quad M = \lim_{t\to\infty} \sup \int_{T(t)}^{t} p(s)ds.$$

In 1983, Ladas et. al [9] obtained an oscillatory criterion

$$L > \frac{1}{e}. \qquad (2 - 5)$$

The condition (2-5) becomes

$$p\tau > \frac{1}{e} \qquad (2 - 6)$$

when $p(t) = p > 0$ and $\tau(t) = \tau > 0$. The condition (2-6) is a single necessary and sufficient condition for all nontrivial solutions of $x'(t) + px(t-\tau) = 0$ to be oscillatory. Also, they showed that (E-5) is oscillatory provided

$$M > 1. \tag{2 - 7}$$

In 1998, Philos and Sficas [12] obtained an oscillatory criteria

$$M + \frac{L^2}{2(1-L)} + \frac{L^2}{2}\lambda_0 > 1 \tag{2 - 8}$$

for (E-5) when $L \leq \frac{1}{e}$. Here $\lambda_0 > 0$ is the small real root of the transcendental equation $\lambda = e^{L\lambda}$. Using the condition (2-8), they proved that (E-5) is oscillatory when the following each case holds :

(i) $L = 0$ and $M > 1$

(ii) $L = \frac{1}{e}$ and $M > 1 - \frac{1}{2(e-1)}$

(iii) $0 < L < \frac{1}{e}$ and $M \geq 1$.

Some examples related in the above results are given in [2].

3. Neutral Delay Differential Equations

We consider the neutral equation

$$\frac{d}{dt}[x(t) + px(t-\tau)] + qx(t-\sigma) = 0, \quad t \geq t_0 \tag{E - 6}$$

where $\tau > 0,\ \sigma > 0,\ p \in \mathbb{R}$ and $q > 0$.

In 1990, Gopalsamy and Zhang [6] showed that if $p = -p_0$ and there exists a positive number μ satisfying

$$p_0 e^{\mu\tau} + \frac{p_0 e^{\mu\sigma}}{\mu} \leq 1, \tag{3 - 1}$$

then (E-6) has a nonoscillatory solution which tends to zero as $t \to \infty$, by using the lemma combining both Banach contraction mapping and the Schauder's fixed point theorem.

Now, we consider the neutral equation

$$\frac{d}{dt}[x(t) - px(t-\tau)] + q(t)x(t-\sigma) = 0, \quad t \geq t_0, \tag{E - 7}$$

where $0 < p < 1,\ \tau > 0,\ \sigma > 0,\ \sigma \geq \tau$, and $q \in C(\mathbb{R}, \mathbb{R}^+)$ is τ-periodic. Gopalsamy and Zhang [6] obtained one sufficient condition for oscillation of (E-7) :

$$Q_0 > \frac{1}{e}(1 - \frac{4p}{Q_0^2}) > 0, \tag{3 - 2}$$

where $Q_0 = \int_{t-\tau}^{t} q(s)ds < \infty$. Also, they showed that (E-7) is oscillatory if

$$q(t) \geq q_0 > 0, \qquad (3 - 3)$$
$$q_0 \sigma e > 1 - p(1 + \frac{\tau q_0}{1-p})$$

by using the Lebesgue's convergence theorem.

In 1991, Ladas et. al [8] imposed the conditions

$$p \in \mathbb{R}, q \in C(\mathbb{R}^+) \text{ is } \omega\text{-periodic with } \omega > 0$$
$$q(t) \not\equiv 0 \text{ for } t \geq 0, \text{ and there exist positive integers}$$
$$n_1 \text{ and } n_2 \text{ such that } \tau = n_1\omega \text{ and } \sigma = n_2\omega, \qquad (3 - 4)$$

for the equation (E-7). Then under the condition (3-4) they showed that (E-7) is oscillatory if and only if an associated neutral equation with constant coefficients $\tau_1, \ \sigma_1$ is oscillatory, where $\tau_1 = \int_0^{\tau} q(s)ds$ and $\sigma_1 = \int_0^{\sigma} q(s)ds$.

Finally, we consider the neutral equation

$$\frac{d}{dt}[x(t) + p(t)x(t-\tau)] + q(t)x(t-\sigma) = 0, \quad t \geq t_0, \qquad (\text{E - 7})$$

where $\tau > 0, \ \sigma \geq 0, \ p \in C([t_0,\infty),\mathbb{R})$ and $q \in C([t_0,\infty),\mathbb{R}^+)$.

Ladas and Sficas [7] obtained a sufficient condition

$$\int_{t_0}^{\infty} q(s)ds = \infty \qquad (3 - 5)$$

for the oscillation of all solutions of (E-7) with $p(t) = -1$. But Wang and Qian [14] had a negative answer about the question that " is condition (3-5) a necessary condition for the oscillation of (E-7) with $p(t) = -1$?". They showed this through an example

$$\frac{d}{dt}[x(t) - x(t-\tau)] + \frac{1}{t^{\alpha}}x(t-\sigma) = 0, \quad 1 < \alpha \leq \frac{3}{2}.$$

Also, they showed that there is a nonoscillatory solution of (E-7) when

$$p(t) = p \neq -1 \text{ and } \int_{t_0}^{\infty} q(s)ds < \infty.$$

Grove et. al [5] assumed that

$$p, \ q \in C([t_0,\infty),\mathbb{R}^+), \ \tau, \ \sigma \in \mathbb{R}^+, \ 0 \leq p(t) \leq 1, t \geq t_0$$
$$0 < K_1 \leq q(t) \leq K_2, \ t \geq t_0 \text{ for some posive constants } K_1, K_2$$
$$\inf_{\mu>0, t\geq T} [\frac{p(t-\sigma)q(t)}{q(t-\tau)}e^{\mu\tau} + \frac{1}{\mu}q(t)e^{\mu\sigma}] > 1, \ T \geq t_0 + m. \qquad (3 - 6)$$

They showed that (E-7) is oscillatory under the conditions (3-6). Their conditions are "sharp" in the sense that when $p(t)$ and $q(t)$ are constants, the conditions become both necessary and sufficient.

4. Problems

We list some problems of oscillation for delay and neutral differential equations
:

(i) Find conditions for all solutions of the equation

$$x'(t) + p(t)x(\tau(t)) = 0,$$

where $\tau(t) \leq t(\not\equiv t)$ and $\lim_{t\to\infty} \tau(t) = \infty$, to be nonoscillatory.

(ii) Study the existence of oscillatory solutions for unstable type delay equation

$$x'(t) = p(t)x(\tau(t)),$$

where $\tau(t) \leq t,\ p(t) \leq 0,\ \lim_{t\to\infty} \tau(t) = \infty$.

(iii) Develop the oscillatory theory for the equation

$$\frac{d}{dt}[x(t) + f(t, x(\tau))] + g(t, x(\sigma)) = 0.$$

(iv) Study oscillation and nonoscillation problems for the equation

$$\frac{d^n}{dt^n}[x(t) + p(t)x(t-\tau)] + q(t)x(\sigma(t)) = 0,$$

with the coefficient $q(t)$, which is not of same sign.

(v) Develop the oscillation theory for the equation

$$\frac{d^n}{dt^n}[x(t) + p(t)f(x(t-\tau))] + q(t)g(x(\sigma(t)) = 0,$$

where f is not almost linear, for example

$$f(x) = |x|^r \text{sign } x,\ r > 0,\ r \neq 1.$$

(vi) Develop the theory for the distribution of zeros of oscillatory solutions.

(vii) Develop the stability theory related to the oscillation theory.

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