Trends in Mathematics
Information Center for Mathematical Sciences
Volume 2, 1999, Pages 000–000

CURVATURE, SPECTRA, AND RIEMANNIAN SUBMERSIONS.

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ABSTRACT. Let Δ_M^p be the Laplacian on p forms on a closed Riemannian manifold M. Let $\pi : Z \rightarrow Y$ be a Riemannian submersion. We give necessary and sufficient conditions to ensure the pull back of every eigen p form over Y is an eigen p form over Z; in this setting eigenvalues can not change so $\mu(\lambda) = \lambda$. We also study the holomorphic and the spinor settings. We show that except in the spin context if a single eigen-section is preserved, then the associated eigenvalue can not decrease. We also show that in many contexts an eigenvalue can change.

§1 INTRODUCTION

Let (M^m, g) be a smooth closed Riemannian manifold. When studying the Dolbeault operator, we shall suppose that M is holomorphic and that g hermitian. When studying the spin context, we shall suppose that M is spin. Let V be a smooth natural vector bundle over M equipped with the natural inner product and connection. For example, we could take $V = \Lambda^p$ to be the bundle of p forms. If we are working in the holomorphic context, then we could take $V = \Lambda^{(p,q)}$ to be the bundle of forms of type (p, q). Finally, if we are working in the spin context, then we could take $V = \mathcal{S}$ to be the complex spinor bundle. Let D_M be the associated operator of Laplace type on $C^\infty(V)$. In the real setting where $V = \Lambda^p$, then $D_M = (d\delta + \delta d)$ is the Laplace-Beltrami operator. In the holomorphic setting where $V = \Lambda^{(p,q)}$, then $D_M = (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)$ is the Dolbeault Laplacian. In the spinor setting where $V = \mathcal{S}$, then $D_M = D_M^{\mathcal{S}}$ is the Spin Laplacian. Let

$$E(\lambda, D_M) := \{\phi \in C^\infty(V) : D_M\phi = \lambda\phi\}$$

be the associated eigenspaces. The $E(\lambda, D_M)$ are finite dimensional and we have an orthogonal direct sum decomposition $L^2(V) = \oplus_\lambda E(\lambda, D_M)$. We say λ is an eigenvalue if $E(\lambda, D_M) \neq \{0\}$; the eigenvalues form a countable discrete set which is bounded from below and which accumulates only at $+\infty$. We refer to [1] for details.

Let $\pi : Z \rightarrow Y$ be a Riemannian submersion. If $V = \Lambda^{(p,q)}$, then we assume that Z and Y are holomorphic manifolds, that the metrics are Hermitian, and that π is a holomorphic map. If $V = \mathcal{S}$, then we assume that Y is spin, that Z is

1991 *Mathematics Subject Classification.* Primary 58G25.

Key words and phrases. Dolbeault Laplacian, Riemannian Submersion, Eigenvalues, Spectra, Bochner Laplacian, Spin Laplacian.

†Research partially supported by the NSF (USA) and MPI (Leipzig, Germany)

‡Research partially supported by Korea Research foundation made in the program year of 1998, and BSRI 98-1425, the Korean Ministry of Education

a principal bundle with compact structure group G over Y, and that Z has the natural induced spin structure; see [4] for details. We then have a natural notion of pull-back $\pi^* : C^\infty(V_Y) \to C^\infty(V_Z)$. In this expository note we want to deal with several questions:

1. What are necessary and sufficient conditions that ensure that $\pi^* E(\lambda, D_Y)$ is a subset of $E(\lambda, D_Z)$ for all λ?
2. If $0 \neq \Phi \in E(\lambda, D_Y)$ and if $\pi^*\Phi \in E(\mu, D_Z)$, what is the relation (if any) between λ and μ?
3. Given λ and μ satisfying the relationship given in (2), can one find a suitable submersion providing examples realizing λ and μ?

This note is divided into several sections. In §2, we discuss the real Laplacian. In §3, we discuss the complex Laplacian. In §4, we discuss the spin Laplacian. We introduce the following notational conventions. We denote the vertical and horizontal distributions of a submersion π by $\mathcal{V} := \ker(\pi_*)$ and $\mathcal{H} := \mathcal{V}^\perp$; π_* is an isometry from $\mathcal{H}_z$ to $T_{\pi z}Y$ for all $z \in Z$.

§2 The real Laplacian

Let $\pi : Z \to Y$ be a Riemannian submersion. We have $\pi^* d_Y = d_Z \pi^*$ by naturality. However, $\pi^*\delta_Y \neq \delta_Z \pi^*$ in general and hence π^* need not intertwine the Laplacians Δ_Y^p and Δ_Z^p. One says that an eigenvalue changes if there exists $0 \neq \Phi \in E(\lambda, \Delta_Y^p)$ so $\pi^*\Phi \in E(\mu, \Delta_Z^p)$ with $\lambda \neq \mu$; this is a comparatively rare phenomena as for a generic Riemannian submersion, the pullback of an eigenform on Y will no longer be an eigenform on Z. Theorems 2.1 and 2.2 below show that eigenvalues can change if $p \geq 2$ and that eigenvalues cannot change if $p = 0$; it is not known if eigenvalues can change if $p = 1$. If pullback preserves all the eigen p forms, then eigenvalues can not change. We refer to [2, 3, 6] for the proof of the following results; see also [8, 13, 14, 18].

2.1 Theorem. *Let $\pi : Z \to Y$ be a Riemannian submersion.*

1. *If $0 \neq \Phi \in E(\lambda, \Delta_Y^p)$ and if $\pi^*\Phi \in E(\mu, \Delta_Z^p)$, then $\lambda \leq \mu$. If $p = 0$, then $\lambda = \mu$.*
2. *Fix p with $0 \leq p \leq \dim_{\mathbb{R}} Y$. The following conditions are equivalent:*
   - i) *$\Delta_Z^p \pi^* = \pi^* \Delta_Y^p$ so $\pi^* E(\lambda, \Delta_Y^p) \subset E(\lambda, \Delta_Z^p)$.*
   - ii) *$\forall \lambda \geq 0, \exists \mu(\lambda) \geq 0$ so $\pi^* E(\lambda, \Delta_Y^p) \subset E(\mu(\lambda), \Delta_Z^p)$.*
   - iii) *The fibers of π are minimal and:*
     - a) *if $p = 0$, there is no further condition.*
     - b) *if $p > 0$, then $\mathcal{H}$ is integrable.*

2.2 Theorem. *Let $0 \leq \lambda < \mu < \infty$ and let $p \geq 2$. There exists a Riemannian submersion $\pi : V \to U$ and there exists $0 \neq \Phi \in E(\lambda, \Delta_U^p)$ so that $\pi^*\Phi \in E(\mu, \Delta_V^p)$.*

There is a natural generalization of Theorem 2.1 to coefficient bundles. Let E be an auxiliary vector bundle over a Riemannian manifold M which is equipped with a smooth inner product $\langle\cdot,\cdot\rangle$ and unitary connection ∇. Such a bundle is said to be a *geometric vector bundle.* In this context, we can define the Bochner Laplacian

$$D(E) := -\operatorname{Tr}(\nabla^*\nabla).$$

Suppose E_Y is given over Y. If $\pi : Z \to Y$ is a Riemannian submersion, we let E_Z be the pull-back bundle with the induced structures. We refer to [7] for the proof of the following result:

2.3 Theorem. *Let $\pi : Z \to Y$ be a Riemannian submersion. Let E_Y be a geometric vector bundle over Y.*

1. *If $0 \neq \Phi \in E(\lambda, D(E_Y))$ and if $\pi^*\Phi \in E(\mu, D(E_Z))$, then $\lambda = \mu$.*
2. *The following conditions are equivalent:*
   i) $D(E_Z)\pi^* = \pi^* D(E_Y)$.
   ii) $\pi^* E(\lambda, D(E_Y)) \subset E(\lambda, D(E_Z))$.
   iii) *The fibers of π are minimal.*

If the boundary of M is non-empty, we must impose suitable boundary conditions. We refer to [1] for a definition of Dirichlet, relative, and absolute boundary conditions. Theorem 2.1 extends to this setting without change for these boundary conditions. We refer to [17] for the proof of the following Theorem:

2.4 Theorem. *Let $\pi : Z \to Y$ be a Riemannian submersion where Y and Z are smooth manifolds with boundary. Let $\mathcal{B}$ denote absolute, relative, or Dirichlet boundary conditions.*

1. *If $0 \neq \Phi \in E(\lambda, \Delta^p_{Y,\mathcal{B}})$ and if $\pi^*\Phi \in E(\mu, \Delta^p_{Z,\mathcal{B}})$, then $\lambda \leq \mu$. If $p = 0$, then $\lambda = \mu$.*
2. *Fix p with $0 \leq p \leq \dim_{\mathbb{R}} Y$. The following conditions are equivalent:*
   i) $\Delta^p_{Z,\mathcal{B}}\pi^* = \pi^*\Delta^p_{Y,\mathcal{B}}$ *so* $\pi^* E(\lambda, \Delta^p_{Y,\mathcal{B}}) \subset E(\lambda, \Delta^p_{Z,\mathcal{B}})$.
   ii) $\forall \lambda \geq 0, \exists \mu(\lambda) \geq 0$ *so* $\pi^* E(\lambda, \Delta^p_{Y,\mathcal{B}}) \subset E(\mu(\lambda), \Delta^p_{Z,\mathcal{B}})$.
   iii) *The fibers of π are minimal and:*
      a) *if $p = 0$, there is no further condition.*
      b) *if $p > 0$, then $\mathcal{H}$ is integrable.*

With Neumann boundary conditions, the situation is quite different. Eigenvalues can be negative and can decrease:

2.5 Theorem. *Let $\mathcal{B}$ denote Neumann boundary conditions. If $p \geq 1$ and if $\mu, \nu \in \mathbb{R}$ are given, there exists a Riemannian submersion $\pi : U \to V$ of compact smooth manifolds with boundary and there exists $0 \neq \Phi \in E(\lambda, \Delta^p_{U,\mathcal{B}})$ so that $\pi^*\Phi \in E(\mu, \Delta^p_{V,\mathcal{B}})$.*

§3 The complex Laplacian

Let $\pi : Z \to Y$ be a Hermitian submersion. We refer to [9, 19] for a discussion of some of the geometry which is involved. We have $\pi^*\bar{\partial}_Y = \bar{\partial}_Z\pi^*$. However, just as in the real case, $\pi^*\bar{\partial}^*_Y \neq \bar{\partial}^*_Z\pi^*$ in general so π^* need not intertwine the complex Laplacians $\Delta^{(p,q)}_Y$ and $\Delta^{(p,q)}_Z$. Theorems 2.1 and 2.2 generalize naturally to the complex category. In Theorem 2.2, we showed eigenvalues could change if the degree was at least 2; in Theorem 3.2, we deal with forms of total degree at least 2 and anti-holomorphic degree at least 1. In the real case, we do not know if a single eigenvalue on 1 forms can change; in the complex case, we do not know if a single eigenvalue on $(0, 1)$ forms can change. Both theorems are incomplete in this respect.

The curvature tensor ω is the obstruction to $\mathcal{H}$ being integrable. Let $\rho_{\mathcal{V}}$ be orthogonal projection on the vertical distribution. If $f_i \in C^\infty\mathcal{H}$, we define

$$\omega(f_1, f_2) := \tfrac{1}{2}\rho_{\mathcal{V}}[f_1, f_2]).$$

Let J be the almost complex structure on the holomorphic manifold Z. We define $J^*\omega(f_1, f_2) := \omega(Jf_1, Jf_2)$. Note that $\omega = 0$ if and only if $\mathcal{H}$ is integrable. Also note that $J^*\omega = \omega$ if and only if the holomorphic horizontal distribution $\mathcal{H}_{(1,0)}$ is integrable. We refer to [5] for the proof of the following two results.

3.1 Theorem. *Let $\pi : Z \to Y$ be a Hermitian submersion.*

1. *If $0 \neq \Phi \in E(\lambda, \Delta_Y^{p,q})$ and if $\pi^*\Phi \in E(\mu, \Delta_Z^{p,q})$, then $\lambda \le \mu$. If $q = 0$, then $\lambda = \mu$.*
2. *Fix (p, q) with $0 \le p, q \le \dim_{\mathbb{C}} Y$. The following conditions are equivalent:*
   i) $\Delta_Z^{p,q}\pi^* = \pi^*\Delta_Y^{p,q}$.
   ii) $\forall\lambda \ge 0, \exists\mu(\lambda) \ge 0$ *so* $\pi^*E(\lambda, \Delta_Y^{p,q}) \subset E(\mu(\lambda), \Delta_Z^{p,q})$.
   iii) *The fibers of π are minimal and:*
      a) *if $p = 0$ and if $q = 0$, then there is no further condition.*
      b) *if $p > 0$ and if $q = 0$, then $J^*\omega = -\omega$.*
      c) *if $p = 0$ and if $q > 0$, then $J^*\omega = \omega$ i.e. $\mathcal{H}_{1,0}$ is integrable.*
      d) *if $p > 0$ and if $q > 0$, then $\mathcal{H}$ is integrable.*

3.2 Theorem. *Let $0 \le \lambda < \mu < \infty$, let $q \ge 1$ and let $p + q \ge 2$. There exists a Hermitian submersion $\pi : V \to U$ and there exists $0 \neq \Phi \in E(\lambda, \Delta_U^{p,q})$ so that $\pi^*\Phi \in E(\mu, \Delta_V^{p,q})$.*

Let E be an auxiliary holomorphic vector bundle over M which is equipped with a unitary metric h. Let $\Delta_E^{(0,q)} := \bar\partial_E\bar\partial_E^* + \bar\partial_E^*\bar\partial_E$ be the Dolbeault Laplacian on $E \otimes \Lambda^{(0,q)}$. We refer to [16] for the proof of the following result:

3.3 Theorem. *Let $\pi : Z \to Y$ be a Hermitian submersion. Let (E_Y, h_Y) be a Hermitian holomorphic bundle over Y.*

1. *If $0 \neq \Phi \in E(\lambda, \Delta_{E_Y}^{(0,q)})$ and if $\pi^*\Phi \in E(\mu, \Delta_{E_Z}^{(0,q)})$, then $\lambda \le \mu$. Furthermore, if $q = 0$, then $\lambda = \mu$.*
2. *Fix q with $0 \le q \le \dim_{\mathbb{C}} Y$. The following conditions are equivalent:*
   (3) i) $\Delta_{E_Z}^{(0,q)}\pi^* = \pi^*\Delta_{E_Y}^{(0,q)}$.
   (4) ii) $\forall\lambda \ge 0, \exists\mu(\lambda) \ge 0$ *so* $\pi^*E(\lambda, \Delta_{E_Y}^{(0,q)}) \subset E(\mu(\lambda), \Delta_{E_Z}^{(0,q)})$.
   (5) iii) *The fibers of π are minimal. If $q = 0$, then there is no further condition. If $q > 0$, then the holomorphic horizontal distribution $\mathcal{H}_{1,0}$ is integrable.*

§4 The spin Laplacian

Let $\pi : P \to Y$ be a principal bundle with compact structure group G. Let Y be a spin manifold. We decompose $TP = TY \oplus \mathfrak{g}$ where $\mathfrak{g}$ is the Lie algebra of G. This decomposition defines a natural spin structure on P so that $\mathcal{S}(P) = \mathcal{S}(Y) \otimes \mathcal{S}(\mathfrak{g})$; here our notation differs a bit from that of Moroianu [10, 11, 12] as we take the associated Clifford bundle rather than the irreducible spinor bundle for the sake of convenience. Fix a spinor $\alpha \in \mathcal{S}(\mathfrak{g})$. We define $\pi^*(\phi) = \phi \otimes \alpha$ to define pull-back; this is the notion of projectable spinors. Let $D^{\mathcal{S}}$ be the spin Laplacian. We refer to [4] for the proof of the following two results:

4.1 Theorem. *Let $\pi : P \to Y$ be a principal bundle. Let $\alpha \in \mathcal{C}\mathfrak{g}$ and let $\epsilon \in \mathbb{R}$. The following conditions are equivalent:*

1. $D_P^{\mathcal{S}}\pi_\alpha^* = \pi_\alpha^*(D_Y^{\mathcal{S}} + \epsilon)$.
2. $\forall\lambda \geq 0$, $\exists\mu(\lambda) \geq 0$ *so* $\pi^* E(\lambda, D_Y^{\mathcal{S}}) \subset E(\mu(\lambda), D_P^{\mathcal{S}})$.
3. *The horizontal distribution defined by π is integrable and $\alpha \in E(\epsilon, D_G^{\mathcal{S}})$.*

4.2 Theorem.

1. *Let G be a compact connected Abelian Lie group. For any $\lambda \geq 0$ and for any $\mu \geq 0$, there exists Y, there exists a principal bundle $\pi : P \to Y$ with structure group G, there exists $0 \neq \Phi \in E(\lambda, D_Y^{\mathcal{S}})$, and there exists $\alpha \in \mathcal{S}(\mathfrak{g})$ such that $\pi_\alpha^*\Phi \in E(\mu, D_P^{\mathcal{S}})$.*
2. *Let G be a compact connected non-Abelian Lie Group. Let $\epsilon_G \geq 0$ be the smallest eigenvalue of $D_G^{\mathcal{S}}$. Let $\lambda \geq 0$ and let μ be given with $\lambda + \epsilon_G \leq \mu$. There exists Y, there exists a principal bundle $\pi : P \to Y$ with structure group G, there exists $0 \neq \Phi \in E(\lambda, D_Y^{\mathcal{S}})$, and there exists $\alpha \in \mathcal{S}\mathfrak{g}$ such that $\pi_\alpha^*\Phi \in E(\mu, D_Y^{\mathcal{S}})$.*

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