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THE SYMPLECTIC STRUCTURE ON THE MODULI SPACE OF REAL PROJECTIVE STRUCTURES

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Abstract. A *convex* real projective structure on a smooth surface M is a representation of M as a quotient of a convex domain $\Omega \subset \mathbb{RP}^2$ by a discrete group $\Gamma \subset \mathbf{PGL}(3,\mathbb{R})$ acting properly and freely on Ω. If $\chi(M) < 0$, then the equivalence classes of convex real projective structures form a moduli space $\mathfrak{P}(M)$ which is an extension of the Teichmüller space $\mathfrak{T}(M)$.

Atiyah and Bott [1] initiated a new approach to the study of moduli spaces of homomorphisms from the fundamental group of a *closed* surface to a compact connected Lie group by methods of the gauge theory. In particular they found natural symplectic structures on certain smooth open subsets of these moduli spaces.

In this paper I will extend the symplectic form to the compact oriented surface with boundary. Since $\mathfrak{P}(M)$ is a Poisson manifold, we find a symplectic foliation called the *parabolic foliation* by giving a restriction on the boundary components. We show the modified form $\tilde{\omega}$ is a symplectic form on each parabolic leaf. We actually calculate the symplectic forms $\tilde{\omega}$ for the pair of pants with using Mathematica.

1. Preliminaries

1.1. Coordinates on the conjugacy classes in $\mathbf{Hyp}_+$. The real projective plane $\mathbb{RP}^2$ is the space of all lines through the origin in $\mathbb{R}^3$. The group of projective transformations of $\mathbb{RP}^2$ is denoted by $\mathbf{PGL}(3,\mathbb{R})$. The homomorphism $\mathbf{GL}(3,\mathbb{R}) \to \mathbf{SL}(3,\mathbb{R})$ defined by

$$A \mapsto (\det A)^{-1/3} A$$

induces an isomorphism $\mathbf{PGL}(3,\mathbb{R}) \to \mathbf{SL}(3,\mathbb{R})$ as *analytic* groups. Thus from now on we shall identify the groups $\mathbf{PGL}(3,\mathbb{R})$ and $\mathbf{SL}(3,\mathbb{R})$.

An element $A \in \mathbf{SL}(3,\mathbb{R})$ is called *positive hyperbolic* if it has three distinct positive real eigenvalues. The set of positive hyperbolic elements of $\mathbf{SL}(3,\mathbb{R})$ is denoted by $\mathbf{Hyp}_+$. If A is positive hyperbolic, then it can be represented by the diagonal matrix

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix} \tag{1}$$

with

$$\lambda\mu\nu = 1, \quad 0 < \lambda < \mu < \nu$$

via an $\mathbf{SL}(3,\mathbb{R})$-conjugation. We define

$$(\lambda, \tau) : \mathbf{Hyp}_+ \to \mathbb{R}^2$$

by $\lambda(A)$ to be the smallest eigenvalue of A and $\tau(A)$ the sum of the other two eigenvalues. For the A above,

$$\lambda(A) = \lambda, \quad \tau(A) = \mu + \nu.$$

As you see the pair $(\lambda(A), \tau(A))$ is invariant under $\mathbf{SL}(3,\mathbb{R})$-conjugation. Then $\{\lambda, \tau\}$ are coordinates on the conjugacy classes in $\mathbf{Hyp}_+$. For more detail see Goldman's paper [5].

1.2. **Convex real projective structures.** A domain $\Omega \subset \mathbb{RP}^2$ is called *convex* if there exist a projective line $l \subset \mathbb{RP}^2$ such that $\Omega \subset (\mathbb{RP}^2 - l)$ and Ω is a convex subset of the affine plane $\mathbb{RP}^2 - l$; i.e. if $x, y \in \Omega$ then the line segment $\bar{xy}$ lies in Ω. By definition, $\mathbb{RP}^2$ itself is not convex. A *convex real projective structure* on a smooth surface M is a diffeomorphism $M \to \Omega/\Gamma$ where $\Omega \subset \mathbb{RP}^2$ is a convex domain and $\Gamma \subset \mathbf{PGL}(3,\mathbb{R})$ is a discrete subgroup acting properly and freely on Ω. A convex $\mathbb{RP}^2$-manifold is a smooth surface with a convex real projective structure.

The following *Development Theorem* is the fundamental fact about convex $\mathbb{RP}^2$-structures.

Theorem 1.1 (Ch. Ehresmann [2]). *Let M be a convex $\mathbb{RP}^2$-manifold and $\tilde{M}$ denotes a universal covering space of M. Let π be the corresponding group of covering transformations.*

1. *There exist a diffeomorphism $\mathbf{dev} : \tilde{M} \to \mathbb{RP}^2$ onto its image Ω and isomorphism $h : \pi \to \mathbf{PGL}(3,\mathbb{R})$ onto its image Γ such that for each $\gamma \in \pi$ the following diagram commutes.*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\mathbf{dev}} & \mathbb{RP}^2 \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle h(\gamma)} \\ \tilde{M} & \xrightarrow[\mathbf{dev}]{} & \mathbb{RP}^2 \end{array}$$

2. *Suppose $(\mathbf{dev}', h')$ is another pair satisfying above condition, then there exists a projective transformation $g \in \mathbf{PGL}(3,\mathbb{R})$ such that $\mathbf{dev}' = g \circ \mathbf{dev}$ and $h' = \iota_g \circ h$ where $\iota_g : \mathbf{PGL}(3,\mathbb{R}) \to \mathbf{PGL}(3,\mathbb{R})$ denotes the inner automorphism defined by g ; i.e. $h'(\gamma) = (\iota_g \circ h)(\gamma) = g \circ h(\gamma) \circ g^{-1}$.*

Proposition 1.2 (N. Kuiper [8]). *Let M be a convex $\mathbb{RP}^2$-manifold. Then each nontrivial element $A \in \Gamma$ is positive hyperbolic.*

Let M be a compact oriented smooth surface. Consider (f, N) where N is a convex $\mathbb{RP}^2$-manifold and $f : M \to N$ a diffeomorphism. Such a pair is equivalent to a developing pair $(\mathbf{dev}, h)$ where $\mathbf{dev} : \tilde{M} \to \mathbb{RP}^2$ and $h : \pi \to \mathbf{PGL}(3,\mathbb{R})$. We say two pairs (f, N) and (f', N') are *equivalent* if there exists a diffeomorphism $g : N \to N'$ such that $g \circ f$ is isotopic to f'. The set of equivalence classes is denote by $\mathfrak{P}(M)$ and is called the *moduli space* of convex $\mathbb{RP}^2$-structures on M.

A *geodesic* on a convex $\mathbb{RP}^2$-manifold M is a curve $g \subset M$ such that for each component $\tilde{g}_0 \subset p^{-1}(g) \subset \tilde{M}$, the developing map takes $\tilde{g}_0$ into a line in $\mathbb{RP}^2$. Suppose M is a convex $\mathbb{RP}^2$-manifold with boundary, then we always assume each boundary component is geodesic.

Theorem 1.3 (W. Goldman). *Let M be a compact oriented surface may having n boundary components with $\chi(M) < 0$. Then $\mathbf{hol} : \mathfrak{P}(M) \to \mathrm{Hom}(\pi, \mathbf{PGL}(3,\mathbb{R}))/\mathbf{PGL}(3,\mathbb{R})$ is an embedding onto a Hausdorff real analytic manifold of dimension $16g - 16 + 8n$.*

1.3. **Tangent space of Hom$(\pi, G)/G$.** Suppose M is a smooth surface with genus g, n boundary components and $\chi(M) = 2-2g-n < 0$. Then $\mathrm{Hom}(\pi, G)$ is an *algebraic variety* where π is the fundamental group of M and $G = \mathbf{SL}(3,\mathbb{R})$. In general $\mathrm{Hom}(\pi, G)$ is not smooth. Suppose $\phi \in \mathrm{Hom}(\pi, G)$ and $Z(\phi)$ is the centralizer of $\phi(\pi)$ in G. Goldman [4] showed ϕ is a nonsingular point of $\mathrm{Hom}(\pi, G)$ if and only if dim $Z(\phi)/Z(G) = 0$ where $Z(G)$ denotes the center of G. Let $\mathrm{Hom}(\pi, G)^-$ be the set of nonsingular points of $\mathrm{Hom}(\pi, G)$. Then G acts freely on the smooth Zariski open subset $\mathrm{Hom}(\pi, G)^-$. But unfortunately $\mathrm{Hom}(\pi, G)^-/G$ is generally not Hausdorff. Let $\mathrm{Hom}(\pi, G)^{--}$ be the subset of $\mathrm{Hom}(\pi, G)^-$ consisting of homomorphisms whose image does not lie in a parabolic subgroup of G. Then $\mathrm{Hom}(\pi, G)^{--}$ is an Zariski open subset of $\mathrm{Hom}(\pi, G)^-$ and $\mathrm{Hom}(\pi, G)^{--}/G$ is a Hausdorff smooth manifold of dimension $-\mathrm{dim}\ G \cdot \chi(M)$. For more detail see Goldman's paper [4].

Suppose $\phi \in \mathrm{Hom}(\pi, G)$ is an irreducible representation. Then $\phi \in \mathrm{Hom}(\pi, G)^{--}$. Since the holonomy representation of a real projective structure from the fundamental group π of a compact oriented Riemann surface to an algebraic Lie group G is irreducible, we restrict our interest to $\mathrm{Hom}(\pi, G)^{--}$.

Let $[\phi] \in \mathrm{Hom}(\pi, G)^{--}/G$ be the equivalence class of $\phi \in \mathrm{Hom}(\pi, G)^{--}$.

Proposition 1.4 (A. Weil [10])**.** *Let π be the fundamental group of a compact oriented surface, G a connected algebraic Lie group. Then the tangent space to* $\mathrm{Hom}(\pi, G)^{--}/G$ *at* $[\phi] \in \mathrm{Hom}(\pi, G)^{--}/G$ *is isomorphic to the first group cohomology* $H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$.

Recall the Lie algebra $\mathfrak{g}$ is a π-module $\mathfrak{g}_{\mathrm{Ad}\phi}$ by the composition

$$\pi \overset{\phi}{\to} G \overset{\mathrm{Ad}}{\to} \mathrm{Aut}(\mathfrak{g}).$$

$u \in Z^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ if and only if $u(xy) = u(x) + \mathrm{Ad}\phi(x)u(y)$ for all $x, y \in \pi$ and $v \in B^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ if and only if there exists $D \in \mathfrak{g}$ such that $v(x) = dD(x) = \mathrm{Ad}\phi(x)D - D$ for all $x \in \pi$.

1.4. **Fox's calculus.** Fox [3] developed a noncommutative differential calculus for words in a free group. Let Π be a free group with generators $x_1, \ldots, x_n$, and $\mathbb{Z}\Pi$ its integral group ring. The *augmentation* homomorphism $\varepsilon : \mathbb{Z}\Pi \to \mathbb{Z}$ is a ring homomorphism defined by $\varepsilon(\sum n_i \sigma_i) = \sum n_i$. The *Fox derivation* of $\mathbb{Z}\Pi$ is a $\mathbb{Z}$-linear map $D : \mathbb{Z}\Pi \to \mathbb{Z}\Pi$ satisfying

$$D(m_1 m_2) = D(m_1)\varepsilon(m_2) + m_1 D(m_2)$$

where $m_1, m_2 \in \mathbb{Z}\Pi$.

Proposition 1.5 (R. Fox [3])**.** *Suppose $x_1, \ldots, x_n$ are the generators for group* Π *and* $\mathrm{Der}(\Pi)$ *is the set of all Fox derivations. Then* $\mathrm{Der}(\Pi)$ *is freely generated as a $\mathbb{Z}\Pi$-module by n element $\partial_i = \partial/\partial x_i, i = 1, \ldots, n$ such that $(\partial/\partial x_i)(x_j) = \delta_{ij} I$ where I is the identity element of* Π.

For any word $w \in \Pi$, computing the Fox derivation ∂w is quite mechanical. For any generators $A, B \in \Pi$, we have

- $\partial_A(AB) = \partial_A(A)\varepsilon(B) + A\partial_A(B) = I$;
- $\partial_B(AB) = \partial_B(A)\varepsilon(B) + A\partial_B(B) = A$;
- $\partial_A(A^{-1}) = -A^{-1}$;
- $\partial_A(ABA^{-1}B^{-1}) = I - ABA^{-1}$;
- $\partial_B(ABA^{-1}B^{-1}) = A - ABA^{-1}B^{-1}$.

¿From the above examples we can see for any word $w \in \Pi$, $\varepsilon(\partial_i w)$ equals the total exponent sum of the letter x_i in the word w.

1.5. **Fundamental cycle.** Consider a closed surface group $\pi = \Pi/R$. Then π is generated by $2g$ generators $A_1, B_1, \dots, A_g, B_g$ and with one relation

$$R = A_1 B_1 A_1^{-1} B_1^{-1} \dots A_g B_g A_g^{-1} B_g^{-1}.$$

Define a 2-chain on π

$$Z_R = \sum_{i=1}^{g} \left((\frac{\partial R}{\partial A_i}, A_i) + (\frac{\partial R}{\partial B_i}, B_i) \right) = \sum_i n_i(x_i, y_i) \in \mathbb{Z}(\pi \times \pi). \tag{2}$$

Then $\partial Z_R = 0$. Recall $\partial : C_2(\Pi) = \mathbb{Z}(\pi \times \pi) \to C_1(\Pi) = \mathbb{Z}(\pi)$ is defined by $\partial(u,v) = \varepsilon(u)v - uv + u\varepsilon(v)$.

R. Lyndon [9] showed the homology class $[Z_R]$ generates $H_2(\pi)$. We call Z_R the *fundamental cycle* of the fundamental group π.

1.6. **The symplectic form on Hom**$(\pi, G)^{--}/G$**.** Let $\mathfrak{g}$ be the Lie algebra of G with an Ad-invariant nondegenerate symmetric bilinear form $\beta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. If $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$, then we could take β is just the trace form. Let $u, v \in Z^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$.

We define a $\mathbb{Z}$-linear map $\beta_*(u,v) : \mathbb{Z}(\pi \times \pi) \to \mathbb{R}$ by

$$\beta_*(u,v)(x,y) = \beta(u(x), \mathrm{Ad}\phi(x) v(y)). \tag{3}$$

Then $\beta_*(u,v) \in Z^2(\pi; \mathbb{R})$.

Theorem 1.6 (Goldman [4])**.** *Let π be the fundamental group of a* closed *surface. Define $\omega : H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \times H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \to \mathbb{R}$ by*

$$\omega([u],[v]) = \beta_*(u,v) Z_R \tag{4}$$

where Z_R is the fundamental cycle (2) of π. Then ω is a symplectic form on $\mathrm{Hom}(\pi, G)^{--}/G$.

2. Moduli space of a surface with boundary

In this section I give an explicit formula for the symplectic structure on a symplectic leaf on the moduli space of real projective structures of a compact oriented surface with boundary.

2.1. **An obstruction for ω to be a symplectic form.** Let M be a compact oriented surface with genus g, n boundary components and $\chi(M) = 2 - 2g - n < 0$. Let π be the fundamental group of M, then π has $2g + n$ generators $A_1, \dots, A_g, B_1, \dots, B_g, C_1, \dots, C_n$ with a single relation $R = \prod_{i=1}^{g} [A_i, B_i] \prod_{j=1}^{n} C_j = I$ where $[A_i, B_i] = A_i B_i A_i^{-1} B_i^{-1}$. We extend the concept of fundamental cycle Z_R of the group π with boundary generators $C_1, \dots, C_n$. Let

$$Z_R = \sum_{k=1}^{g} \left(\frac{\partial R}{\partial A_k}, A_k \right) + \sum_{k=1}^{g} \left(\frac{\partial R}{\partial B_k}, B_k \right) + \sum_{h=1}^{n} \left(\frac{\partial R}{\partial C_h}, C_h \right). \tag{5}$$

Then $\partial Z_R = \sum_{j=1}^{n} C_j$. So the boundary of Z_R is the sum of boundary generators. Hence we will call Z_R the *fundamental relative cycle* of π.

We define a bilinear form $\omega : Z^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \times Z^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \to \mathbb{R}$ by

$$\omega(u,v) = \beta_*(u,v) Z_R$$

where Z_R is the fundamental relative cycle (5) of π and β_* is defined as before in (3).

In the following proposition 2.1, we first meet an obstruction for ω to be symplectic. If π has boundary generators $C_1, \ldots, C_n$, then ω is not skew-symmetric

Proposition 2.1. *For any $u, v \in Z^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$,*

$$\omega(u,v) + \omega(v,u) + \sum_{j=1}^{n} \beta(u(C_j), v(C_j)) = 0$$

where $C_1, \ldots, C_n$ are the boundary generators of π.

2.2. **Parabolic cohomology.** The boundary ∂M of M is a disjoint union of $(\partial M)_j,\ j = 1, \ldots, n$ such that each boundary component is diffeomorphic to the unit circle S^1. Let π_j be the fundamental group of $(\partial M)_j$. Then for each $j,\ \pi_j$ is the cyclic group $< C_j >$ generated by C_j. Define

$$\begin{aligned} C^p(\{\pi_j\}; \mathfrak{g}_{\mathrm{Ad}\phi}) &= \bigoplus_{j=1}^{n} C^p(\pi_j; \mathfrak{g}_{\mathrm{Ad}\phi}) \\ &= \{(f_1, \ldots, f_n) : \pi_1^p \times \cdots \times \pi_n^p \to \mathfrak{g}_{\mathrm{Ad}\phi}^p\} \end{aligned}$$

and $d : C^p(\{\pi_j\}; \mathfrak{g}_{\mathrm{Ad}\phi}) \to C^{p+1}(\{\pi_j\}; \mathfrak{g}_{\mathrm{Ad}\phi})$ by $d(f_1, \ldots, f_n) = (df_1, \ldots, df_n)$. Then $d \circ d = 0$. Therefore $\{C^p(\{\pi_j\}; \mathfrak{g}_{\mathrm{Ad}\phi}), d\}$ forms a cochain complex.

We define $\alpha : C^p(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \to C^p(\{\pi_j\}; \mathfrak{g}_{\mathrm{Ad}\phi})$ by $\alpha(f) = (f_1, \ldots, f_n)$ where $f_j = f|_{\pi_j} : \pi_j^p \to \mathfrak{g}_{\mathrm{Ad}\phi}$. Then the following diagram commutes.

$$\begin{array}{ccc} C^{p-1}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) & \xrightarrow{\alpha} & C^{p-1}(\{\pi_j\}; \mathfrak{g}_{\mathrm{Ad}\phi}) \\ d\downarrow & & \downarrow d \\ C^p(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) & \xrightarrow[\alpha]{} & C^p(\{\pi_j\}; \mathfrak{g}_{\mathrm{Ad}\phi}) \end{array}$$

We define $Z^p_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) = \{f \in Z^p(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \mid \alpha(f) \in B^p(\{\pi_j\}; \mathfrak{g}_{\mathrm{Ad}\phi})\}$. Then we have the following relation.

$$B^p(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \subseteq Z^p_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \subseteq Z^p(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}). \tag{6}$$

We define

$$H^p_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) = \frac{Z^p_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})}{B^p(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})}.$$

Then we have a long exact sequence

$$\begin{aligned} \cdots \to H^{p-1}_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) &\to H^{p-1}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \to H^{p-1}(\{\pi_j\}; \mathfrak{g}_{\mathrm{Ad}\phi}) \\ \to H^p_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) &\to H^p(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \to H^p(\{\pi_j\}; \mathfrak{g}_{\mathrm{Ad}\phi}) \to \cdots . \end{aligned}$$

Since $H^0(\{\pi_j\}; \mathfrak{g}_{\mathrm{Ad}\phi}) = 0$, we get a relation

$$H^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \subseteq H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}).$$

Suppose M is closed. Then $H^p(\{\pi_j\}; \mathfrak{g}_{\mathrm{Ad}\phi}) = 0$ for all p. That means $H^p_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ is the same as $H^p(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ if M is a closed surface. For more detail about the parabolic cohomology, see Guruprasad, Huebschmann, Jeffrey, and Weinstein's joint paper [6].

Consider $p = 1$ (since we are interested in $H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ the tangent space of $\mathrm{Hom}(\pi, G)^{--}/G$). Let $u \in Z^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$, then $u \in Z^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ and $\alpha(u) \in B^1(\{\pi_j\}; \mathfrak{g}_{\mathrm{Ad}\phi})$.

That means $u(xy) = u(x) + x.u(y)$ for $x, y \in \pi$ and there exist $D_1, \dots, D_n \in C^0(\pi_j; \mathfrak{g}_{\mathrm{Ad}\phi}) = \mathfrak{g}$ such that $u_j = dD_j$ where $u_j = u|_{\pi_j}$. Since $\pi_j =< C_j >$,

$$u(C_j) = u_j(C_j) = dD_j(C_j) = C_j.D_j - D_j. \tag{7}$$

A. Weil [10] called the 1-cocycle satisfying the condition (7) a *parabolic element*. We will use his terminology.

2.3. **The extended symplectic form.** Let D be an element of the Lie algebra $\mathfrak{g} = C^0(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$, then $dD \in B^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ such that $dD(x) = x.D - D$ for any $x \in \pi$.

Proposition 2.2. *For any $u, v \in Z^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ and $D \in \mathfrak{g}$,*

$$\omega(dD, v) = \sum_{j=1}^{n} \beta(D, v(C_j)), \quad \omega(u, dD) = \sum_{j=1}^{n} \beta(u(C_j^{-1}), D). \tag{8}$$

We know ω is not skew-symmetric on $Z^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ if M has nonempty boundary. To make ω skew-symmetric, we restrict our domain to the subspace $Z^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \subseteq Z^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ and define a bilinear form

$$\tilde{\omega} : Z^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \times Z^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \to \mathbb{R} \ \text{ by}$$

$$\tilde{\omega}(u, v) = \omega(u, v) - \sum_{j=1}^{n} \beta(D_j, v(C_j))$$

where $u|_{\pi_j} = u_j = dD_j$.

Suppose we have another $\bar{D}_j \in \mathfrak{g}$ such that $u|_{\pi_j} = dD_j = d\bar{D}_j$. Then we can show

$$\beta(\bar{D}_j, v(C_j)) = \beta(D_j, v(C_j)).$$

So $\sum_{j=1}^{n} \beta(D_j, v(C_j))$ does not depend on the choices of D_j's. Therefore $\tilde{\omega}$ is well-defined on $Z^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$. If M is closed, then $\tilde{\omega} = \omega$.

Proposition 2.3. *$\tilde{\omega}$ is skew-symmetric on $Z^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$.*

Consider $dD \in B^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \subseteq Z^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$. Then $dD|_{\pi_j} = dD$ for all j. By Equation (8) and skew-symmetry, for any $v \in Z^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$

$$\begin{aligned} \tilde{\omega}(dD, v) &= \omega(dD, v) - \sum_{j=1}^{n} \beta(D, v(C_j)) = 0 \\ \tilde{\omega}(v, dD) &= -\tilde{\omega}(dD, v) = 0 \end{aligned}$$

Therefore we can induce $\tilde{\omega}$ on $H^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$.

Theorem 2.4. *Define $\tilde{\omega} : H^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \times H^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \to \mathbb{R}$ by*

$$\tilde{\omega}([u], [v]) = \tilde{\omega}(u, v) = \omega(u, v) - \sum_{j=1}^{n} \beta(D_j, v(C_j))$$

where $u, v \in Z^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ representing $[u], [v] \in H^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ and $u|_{\pi_j} = D_j$. Then $\tilde{\omega}$ is a symplectic form on $H^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$.

Proof. See Hong C. Kim's dissertation [7]. □

2.4. **A necessary and sufficient condition for u_{z_k} to be a parabolic element.** Let $\{z_1, \ldots, z_N\}$ be a coordinates of $\mathfrak{P}(M)$ where $N = 16g - 16 + 8n$. Since the holonomy homomorphism $h : \pi \to G = \mathbf{SL}(3, \mathbb{R})$ is isomorphic to its image holonomy group Γ, we may identify the elements of π and those image in G up to conjugate. Then $A.X = AXA^{-1}$ for $A \in \pi$ and $X \in \mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$.

For each k, we define $u_{z_k} : \pi \to \mathfrak{g}$ by

$$u_{z_k}(A) = (\partial_{z_k} A)A^{-1}$$

since $\mathrm{tr}\{(\partial_{z_k} A)A^{-1}\} = \partial_{z_k}\{\log \det A\} = 0$.

Then for any $A, B \in \pi$,

$$\begin{aligned} u_{z_k}(AB) &= (\partial_{z_k} AB)(AB)^{-1} = \{(\partial_{z_k} A)B + A(\partial_{z_k} B)\}B^{-1}A^{-1} \\ &= (\partial_{z_k} A)A^{-1} + A(\partial_{z_k} B)B^{-1}A^{-1} = u_{z_k}(A) + A.u_{z_k}(B). \end{aligned}$$

Therefore $u_{z_k} \in Z^1(\pi; \mathfrak{g})$ for each k.

We want to find a necessary and sufficient condition for u_{z_k} to be a parabolic element. Since $\mathfrak{P}(M)$ is the moduli space of convex real projective structures, every element of π has three distinct positive real eigenvalues. For each boundary generator C_j there exists an invertible matrix $P_j \in G$ such that $P_j^{-1} C_j P_j = E_j$ the diagonal matrix (1).

Proposition 2.5. *Suppose z_k is a coordinate in $\{z_1, \ldots, z_N\}$. Then u_{z_k} is a parabolic element if and only if $\partial_{z_k}(P_j^{-1} C_j P_j) = \partial_{z_k}(E_j) = 0$ for each boundary generator C_j where $j = 1, \ldots, n$.*

Proof. Suppose $u_{z_k}(C) = dD(C)$. Then

$$(\partial_{z_k} C)C^{-1} = u_{z_k}(C) = dD(C) = C.D - D = CDC^{-1} - D.$$

Therefore $(\partial_{z_k} C) = CD - DC$. That means u_{z_k} is a parabolic element if and only if there exist $D_1, \ldots, D_n \in \mathfrak{g}$ such that $\partial_{z_k} C_j = C_j D_j - D_j C_j$ for each $j = 1, \ldots, n$. For the simplicity put $C_j = C,\ E_j = E,$ and $P_j = P$. Since $E = P^{-1}CP$,
$\partial_{z_k} C = \partial_{z_k}(PEP^{-1}) = (\partial_{z_k} P)EP^{-1} + P(\partial_{z_k} E)P^{-1} + PE(\partial_{z_k} P^{-1})$
$= (\partial_{z_k} P)(P^{-1}CP)P^{-1} + P(\partial_{z_k} E)P^{-1} + P(P^{-1}CP)(-P^{-1}(\partial_{z_k} P)P^{-1})$
$= (\partial_{z_k} P)P^{-1}C + P(\partial_{z_k} E)P^{-1} - C(\partial_{z_k} P)P^{-1}$.
Let $D = -(\partial_{z_k} P)P^{-1}$. Then $\partial_{z_k} C = CD - DC$ if and only if $\partial_{z_k} E = 0$. ◻

Now we can calculate the dimension of $H^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$.

$$\dim\ H^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) = \dim\ H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) - 2n = 16g - 16 + 6n. \tag{9}$$

3. Example : Pair of pants $\Sigma(0,3)$

Let M be a pair of pants $\Sigma(0,3)$. Then π admits a representation $< C_1, C_2, C_3 \mid R = C_1C_2C_3 = I >$ and dim $\mathfrak{P}(M) = 16g - 16 + 8n = 8$. Since each positive hyperbolic element C_j has two parameters (λ_j, τ_j) and π has three generators, we need two more parameters s, t. We call s, t the *internal parameters* of $\mathfrak{P}(M)$. Goldman [5] expressed C_1, C_2, C_3 using parameters $\{\lambda_1,\ \lambda_2,\ \lambda_3,\ \tau_1,\ \tau_2,\ \tau_3,\ s,\ t\}$ of $\mathfrak{P}(M)$ where $0 < \lambda_j < 1,\ 2/\sqrt{\lambda_j} < \tau_j < \lambda_j + \lambda_j^{-2},\ s > 0$ and $t > 0$. Since s and t are positive, we replace the coordinates s and t by e^s and e^t. Then the matrices C_1, C_2 and C_3

are represented by

$$C_1 = \begin{pmatrix} \frac{e^{-s}\sqrt{\lambda_3}}{\sqrt{\lambda_2}\sqrt{\lambda_1}} + \tau_1 & \frac{e^{-s+t}\sqrt{\lambda_3}}{\sqrt{\lambda_2}\sqrt{\lambda_1}} & 0 \\ -\frac{e^{-s-t}\sqrt{\lambda_3}}{\sqrt{\lambda_2}\sqrt{\lambda_1}} - \frac{e^{s-t}\sqrt{\lambda_2}}{\sqrt{\lambda_3}\sqrt{\lambda_1}} - e^{-t}\tau_1 & -\frac{e^{-s}\sqrt{\lambda_3}}{\sqrt{\lambda_2}\sqrt{\lambda_1}} & 0 \\ a_{31} & a_{32} & \lambda_1 \end{pmatrix}$$

where $a_{31} = \frac{1}{2}e^{2s}\lambda_3 + \frac{e^{-s-t}\sqrt{\lambda_3}}{2\sqrt{\lambda_2}\sqrt{\lambda_1}} + \frac{e^{s-t}\sqrt{\lambda_2}}{2\sqrt{\lambda_3}\sqrt{\lambda_1}} + \frac{e^{s-t}\sqrt{\lambda_3}\sqrt{\lambda_1}}{2\ \lambda_2^{3/2}} + \frac{e^{3s-t}\sqrt{\lambda_1}}{2\sqrt{\lambda_3}\sqrt{\lambda_2}} + \frac{\lambda_1}{2} + \frac{1}{2}e^{2s-t}\tau_3 + \frac{e^{-t}\lambda_3\tau_3}{2\lambda_2} + \frac{1}{2}e^{s}\sqrt{\lambda_3}\sqrt{\lambda_2}\sqrt{\lambda_1}\tau_2 + \frac{1}{2}e^{-t}\tau_1 + \frac{e^{2s-t}\lambda_1\tau_1}{2\lambda_2} + \frac{e^{s-t}\sqrt{\lambda_3}\sqrt{\lambda_1}\tau_3\tau_1}{2\sqrt{\lambda_2}}$ and $a_{32} = \frac{e^{-s}\sqrt{\lambda_3}}{2\sqrt{\lambda_2}\sqrt{\lambda_1}} + \frac{e^{s}\sqrt{\lambda_3}\sqrt{\lambda_1}}{2\ \lambda_2^{3/2}} + \frac{\lambda_3\tau_3}{2\lambda_2}$,

$$C_2 = \begin{pmatrix} -\frac{e^{-s}\sqrt{\lambda_1}}{\sqrt{\lambda_3}\sqrt{\lambda_2}} & 0 & -\frac{2e^{-s}\sqrt{\lambda_1}}{\sqrt{\lambda_3}\sqrt{\lambda_2}} \\ \frac{e^{-s-t}\sqrt{\lambda_1}}{\sqrt{\lambda_3}\sqrt{\lambda_2}} + \frac{\sqrt{\lambda_2}\sqrt{\lambda_1}}{e^{t-s}\lambda_3^{3/2}} + \frac{e^{-t}\lambda_1\tau_1}{\lambda_3} & \lambda_2 & b_{23} \\ \frac{e^{s}\sqrt{\lambda_3}}{2\sqrt{\lambda_2}\sqrt{\lambda_1}} + \frac{e^{-s}\sqrt{\lambda_1}}{2\sqrt{\lambda_3}\sqrt{\lambda_2}} + \frac{\tau_2}{2} & 0 & \frac{e^{-s}\sqrt{\lambda_1}}{\sqrt{\lambda_3}\sqrt{\lambda_2}} + \tau_2 \end{pmatrix}$$

where $b_{23} = 2\lambda_2 + \frac{2e^{-s-t}\sqrt{\lambda_1}}{\sqrt{\lambda_3}\sqrt{\lambda_2}} + \frac{2\sqrt{\lambda_2}\sqrt{\lambda_1}}{e^{t-s}\lambda_3^{3/2}} + \frac{2\lambda_1\tau_1}{e^{t}\lambda_3}$,

$$C_3 = \begin{pmatrix} \lambda_3 & e^{t}\lambda_3 + \frac{e^{-s}\sqrt{\lambda_2}}{\sqrt{\lambda_3}\sqrt{\lambda_1}} + \frac{e^{s}\sqrt{\lambda_1}}{\sqrt{\lambda_3}\sqrt{\lambda_2}} + \tau_3 & \frac{2e^{-s}\sqrt{\lambda_2}}{\sqrt{\lambda_3}\sqrt{\lambda_1}} \\ 0 & \frac{e^{-s}\sqrt{\lambda_2}}{\sqrt{\lambda_3}\sqrt{\lambda_1}} + \tau_3 & \frac{2e^{-s}\sqrt{\lambda_2}}{\sqrt{\lambda_3}\sqrt{\lambda_1}} \\ 0 & -\frac{e^{-s}\sqrt{\lambda_2}}{2\sqrt{\lambda_3}\sqrt{\lambda_1}} - \frac{e^{s}\sqrt{\lambda_1}}{2\sqrt{\lambda_3}\sqrt{\lambda_2}} - \frac{\tau_3}{2} & -\frac{e^{-s}\sqrt{\lambda_2}}{\sqrt{\lambda_3}\sqrt{\lambda_1}} \end{pmatrix}$$

up to conjugation. For more detail see Goldman [5]. The eigenvalues λ_j, μ_j, ν_j of C_j are as follow.

$$\left\{\lambda_1,\ \mu_1 = \frac{\sqrt{\lambda_1}\ \tau_1 - \sqrt{-4 + \lambda_1\ \tau_1^2}}{2\ \sqrt{\lambda_1}},\ \nu_1 = \frac{\sqrt{\lambda_1}\ \tau_1 + \sqrt{-4 + \lambda_1\ \tau_1^2}}{2\ \sqrt{\lambda_1}}\right\},$$

$$\left\{\lambda_2,\ \mu_2 = \frac{\sqrt{\lambda_2}\ \tau_2 - \sqrt{-4 + \lambda_2\ \tau_2^2}}{2\ \sqrt{\lambda_2}},\ \nu_2 = \frac{\sqrt{\lambda_2}\ \tau_2 + \sqrt{-4 + \lambda_2\ \tau_2^2}}{2\ \sqrt{\lambda_2}}\right\},$$

$$\left\{\lambda_3,\ \mu_3 = \frac{\sqrt{\lambda_3}\ \tau_3 - \sqrt{-4 + \lambda_3\ \tau_3^2}}{2\ \sqrt{\lambda_3}},\ \nu_3 = \frac{\sqrt{\lambda_3}\ \tau_3 + \sqrt{-4 + \lambda_3\ \tau_3^2}}{2\ \sqrt{\lambda_3}}\right\}.$$

¿From (9), dim $H^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) = 8 - 6 = 2$. By Proposition 2.5, u_s and u_t are parabolic elements since eigenvalues of C_1, C_2 and C_3 don't involve s and t.

Now we calculate the symplectic form $\tilde{\omega}$. Let's compute $\tilde{\omega}(u_s, u_t)$. Suppose $u_s|_{\pi_1} = dD_1, u_s|_{\pi_2} = dD_2$ and $u_s|_{\pi_3} = dD_3$. Since $Z_R = (\partial_{C_1}R, C_1) + (\partial_{C_2}R, C_2) + (\partial_{C_3}R, C_3) = (I, C_1) + (C_1, C_2) + (C_1C_2, C_3)$,

$$\begin{aligned}
&\omega(u_s, u_t) = \beta_*(u_s, u_t)Z_R = \beta(u_s(I), I.u_t(C_1)) \\
&+\beta(u_s(C_1), C_1.u_t(C_2)) + \beta(u_s(C_1C_2), C_1C_2.u_t(C_3)) \\
&= 0 + \beta(u_s(C_1), C_1.u_t(C_2)) + \beta(u_s(C_3^{-1}), C_3^{-1}.u_t(C_3)) \\
&= \beta(u_s(C_1), C_1.u_t(C_2)) + \beta(C_3.u_s(C_3^{-1}), u_t(C_3)) \\
&= \beta(u_s(C_1), C_1u_t(C_2)C_1^{-1}) - \beta(u_s(C_3)), u_t(C_3)) \\
&= \mathrm{tr}\{(\partial_sC_1)C_1^{-1}C_1(\partial_tC_2)C_2^{-1}C_1^{-1}\} - \mathrm{tr}\{(\partial_sC_3)C_3^{-1}(\partial_tC_3)C_3^{-1}\} \\
&= \mathrm{tr}\{(\partial_sC_1)(\partial_tC_2)C_3 + (\partial_sC_3)(\partial_tC_3^{-1})\}.
\end{aligned}$$

So $\tilde{\omega}(u_s, u_t)$

$$\begin{aligned} &= \omega(u_s, u_t) - \beta(D_1, u_t(C_1)) - \beta(D_2, u_t(C_2)) - \beta(D_3, u_t(C_3)) \\ &= \operatorname{tr}\{(\partial_s C_1)(\partial_t C_2)C_3 + (\partial_s C_3)(\partial_t C_3^{-1}) \\ &\qquad - D_1(\partial_t C_1)C_1^{-1} - D_2(\partial_t C_2)C_2^{-1} - D_3(\partial_t C_3)C_3^{-1}\}. \end{aligned}$$

Let P_1 be a matrix such that $P_1^{-1}C_1P_1$ is diagonal. Then by the proof of Proposition 2.5, $-(\partial_s P_1)P_1^{-1}$ is one of D_1. Therefore we have the explicit formula for symplectic form $\tilde{\omega}(u_s, u_t)$

$$\begin{aligned} &= \operatorname{tr}\{(\partial_s C_1)(\partial_t C_2)C_3 + (\partial_s C_3)(\partial_t C_3^{-1}) + (\partial_s P_1)P_1^{-1}(\partial_t C_1)C_1^{-1} \\ &\qquad + (\partial_s P_2)P_2^{-1}(\partial_t C_2)C_2^{-1} + (\partial_s P_3)P_3^{-1}(\partial_t C_3)C_3^{-1}\}. \end{aligned}$$

Put P_j the transpose of an eigenvector matrix of C_j for each j. Then we get an amazing result calculated by Mathematica.

$$\tilde{\omega}(u_s, u_t) = -1.$$

Theorem 3.1. *Suppose M is a pair of pants $\Sigma(0,3)$ and $\mathfrak{P}(M)$ is the moduli space of convex real projective structures on M. Then the symplectic form on each parabolic leaf of $\mathfrak{P}(M)$ is*

$$\tilde{\omega} = dt \wedge ds.$$

Proof. See the detailed Mathematica computation in the appendix of Hong C. Kim's dissertation [7]. □

References

[1] M. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Phil. Trans. R. Soc. London A **308** (1982), 523–615.

[2] Ch. Ehresmann, *Sur les espaces localement homogenes*, L'ens Math. **35** (1936), 317–333.

[3] R. Fox, *The free differential calculus, I. Derivations in the group ring*, Ann. of Math. **57** (1953), 547–560.

[4] W. Goldman, *The symplectic nature of fundamental groups of surfaces*, Advances in Mathematics **54** (1984), 200–225.

[5] ———, *Convex real projective structures on compact surfaces*, J. Differential Geometry **31** (1990), 126–159.

[6] K. Guruprasad, J. Huebschmann, L. Jeffrey, and A. Weinstein *Group systems, groupoids, and moduli spaces of parabolic bundles*, Duke Math. J. **89** (1997), no. 2, 377–412.

[7] H. Kim, *The symplectic structure on the moduli space of real projective structures*, Ph.D. dissertation, University of Maryland, 1999, 1–90.

[8] N. Kuiper, *On convex locally projective spaces*, Convegno Int. Geometria Diff., Italy, 1954, 200–213.

[9] R. Lyndon, *Cohomology theory of groups with one defining relation*, Ann. of Math. **52** (1950), 650–665.

[10] A. Weil, *Remark on the Cohomology of groups*, Ann. of Math. **80** (1964), 149–157.

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