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FINITE ADDITIVE MEASURE AND HYPERBOLIC SPACE

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Abstract. We can extend the hyperbolic space beyond the infinity of the space by finite additive measure theory. This extended space which contains hyperbolic space as a subset has many natural and surprising properties which are similar to the case of spherical geometry.

0. Introduction

There are many hyperbolic models representing hyperbolic space $\mathbb{H}^n$, for example, upper half space model, unit disk model, hyperboloid model and Kleinian model. The Kleinian model K^n which is our matter of concern has the straight lines as geodesics. Here the Kleinian model is only defined at the inside of an unit ball by a well known Riemannian metric

$$ds^2 = \left(\frac{\Sigma x_i dx_i}{1-|x|^2}\right)^2 + \frac{\Sigma dx_i^2}{1-|x|^2}.$$

This Riemannian metric induces a volume form

$$dV = \frac{dx_1 \wedge \cdots \wedge dx_n}{(1-|x|^2)^{\frac{n+1}{2}}}.$$

Here we can reform the Kleinian metric to a Kleinian ϵ-metric

$$ds_\epsilon^2 = \left(\frac{\Sigma x_i dx_i}{d_\epsilon^2-|x|^2}\right)^2 + \frac{\Sigma dx_i^2}{d_\epsilon^2-|x|^2},$$

where $d_\epsilon = 1-\epsilon i$, then the Kleinian volume form is changed to a Kleinian ϵ-volume form

$$dV_\epsilon = \frac{d_\epsilon \; dx_1 \wedge \cdots \wedge dx_n}{(d_\epsilon^2-|x|^2)^{\frac{n+1}{2}}}.$$

In the Kleinian model, the volume is calculated by

$$\mu(U) = \int_U \frac{dx_1 \wedge \cdots \wedge dx_n}{(1-|x|^2)^{\frac{n+1}{2}}}, \text{ for a set } U \text{ in } K^n.$$

By the way the Kleinian ϵ-volume form induces a volume for any set U in $\mathbb{R}^n$ which contains K^n as a subset by

$$(1) \qquad \overline{\mu}(U) = \lim_{\epsilon\to 0} \int_U \frac{d_\epsilon \; dx_1 \wedge \cdots \wedge dx_n}{(d_\epsilon^2-|x|^2)^{\frac{n+1}{2}}},$$

and the above two volume integrals coincide at the case that a set U is contained in K^n. So the second integral volume formula (1) is the generalization of the first integral formula. Particularly for a set which intersects transversally to the ideal boundary of the hyperbolic space, the volume of the set has a finite complex value.

The volumes on the extended hyperbolic space have the hyperbolic invariant property as well as the hyperbolic space, i.e., $\overline{\mu}(f(U)) = \overline{\mu}(U)$ for an arbitrary hyperbolic isometry f. However the new measure $\overline{\mu}$ does not admit the countable additive property. So we need find a suitable measure theory for this case: finite additive measure theory with an algebra (change countable union property to finite one) and a finite additive measure. Though the finite additive measure is weaker than an usual measure, we can construct a sufficient and natural theory on the extended Kleinian model $\overline{K}^n$.

The distance between two points on the extended space was firstly defined by Schlenker [4] by cross ratio. There he considered only distance. But our ϵ-technique can be adapted for any other geometric objects, for example, distance between two points (easily turned out to be same to the definition of Schlenker), lengths of a curves, angles, k-dimensional volumes and so on (see [1]).

1. Finite additive invariant measure on the extended hyperbolic space

In order to speak about finite additive measure theory on the extended space, we have to construct $(\overline{K}^n, \mathcal{M}, \mu)$, where $\mathcal{M}$ is an algebra in $\overline{K}^n$ and μ is a finite additive measure defined on $\mathcal{M}$.

The finite additive measure μ is already chosen as $\overline{\mu}$ in (1). Hence the remaining parts are a construction of $\mathcal{M}$ and the finite additivity of $\overline{\mu}$ on $\mathcal{M}$.

Before the construction, let's define the four Borel subcollections $\mathbf{U}_1, \mathbf{U}_2$, $\mathbf{U}_3$, and $\mathbf{U}_4$ on $\overline{K}^n$ and represent an element of $\mathbf{U}_i$ as U_i. $\mathbf{U}_1$ and $\mathbf{U}_2$ is a collection of Borel sets in K^n and $\overline{K}^n \setminus K^n$, respectively, with finite volume, $\mathbf{U}_3$ is a collection of center cones whose vertices are origin with Borel set sections, and $\mathbf{U}_4$ is a collection of $U_3 \setminus (U_1 \cup U_2)$.

And let $\mathcal{M}$ be a smallest algebra generated by $\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3$ types, and $\mathcal{M}'$ be a set of $U_1 \cup U_2 \cup U_4$, and $\mathcal{M}''$ be a set of $U_1 \dot{\cup} U_2 \dot{\cup} U_4$. Then we have $\mathcal{M} = \mathcal{M}' = \mathcal{M}''$ (see [2]), so we can imagine the geometric figures for members of $\mathcal{M}$. The algebra $\mathcal{M}$ contains all Borel sets which intersect transversally to the ideal boundary of the hyperbolic space, and all members of $\mathcal{M}$ have a finite volume (see [2]). The proof of finite additivity of $\overline{\mu}$ on $\mathcal{M}$ is also easy.

For disjoint $o_i \in \mathcal{M}$, we get

$$\begin{aligned}\overline{\mu}(o_1 \cup o_2) &= \lim_{\epsilon \to 0} \int_{o_1 \cup o_2} dV_\epsilon = \lim_{\epsilon \to 0} (\int_{o_1} dV_\epsilon + \int_{o_2} dV_\epsilon) \\ &= \lim_{\epsilon \to 0} \int_{o_1} dV_\epsilon + \lim_{\epsilon \to 0} \int_{o_2} dV_\epsilon = \overline{\mu}(o_1) + \overline{\mu}(o_2).\end{aligned}$$

For a center cone U_3 with a radius R and a section B, the volume of U_3 is obtained by a different and useful method (see [2])

$$\begin{aligned}\overline{\mu}(U_3) &= \lim_{\epsilon\to 0}\int_{U_3}\frac{d_\epsilon\ dx_1\wedge\cdots\wedge dx_n}{(d_\epsilon^2-|x|^2)^{\frac{n+1}{2}}}\\ &= \lim_{\epsilon\to 0}\int_B\int_0^R\frac{d_\epsilon r^{n-1}}{(d_\epsilon^2-r^2)^{\frac{n+1}{2}}}dr\ d\theta\\ &= \int_B\int_\gamma\frac{r^{n-1}}{(1-r^2)^{\frac{n+1}{2}}}dr\ d\theta,\end{aligned}$$

where $d\theta$ is the volume form of the Euclidean unit sphere $\mathbb{S}^{n-1}$ and γ represents a contour from 0 to R in $\mathbb{C}$ traversing in a clockwise direction around 1, i.e., γ is an over-going path with respect to 1.

A hyperbolic sphere model $\mathbb{S}^n_H$ which is topologically equivalent to an unit sphere and considered a two fold covering of $\overline{K}^n$ is better than the extended Kleinian model $\overline{K}^n$, when we consider invariance. For any transformation f in Lorentz group $O(n,1)$ which contains the hyperbolic isometry group as an index two subgroup and any set U in $\mathcal{M}$, we obtain $\overline{\mu}(f(U)) = \overline{\mu}(U)$. This invariance is an important property and difficult to prove (see [2]).

2. Extended hyperbolic space

In this section, we study the hyperbolic sphere model $\mathbb{S}^n_H$ and the extended Kleinian model $\overline{K}^n$ as an analytic continuation of the hyperbolic space $\mathbb{H}^n$. The hyperbolic sphere $\mathbb{S}^n_H$ is topologically a sphere S^n which is obtained by projectivizing $\mathbb{R}^{n,1}$ with only $\mathbb{R}^+$, the positive real numbers, we identifying a half lay emanating from the origin to a point. Hence $\mathbb{S}^n_H$ is a double cover of $\mathbb{R}P^n$ and consists of three parts: the hyperbolic part H^n, the Lorentzian part S^n_1 and the projectivization of the light cone.

The metric on $\mathbb{S}^n_H$ is given by the induced metric on H^n for the hyperbolic part H^n and the negative of the induced metric on the Lorentzian part S^n_1. Sometime, it would be helpful if we visualize $\mathbb{S}^n_H$ as the Euclidean sphere in $\mathbb{R}^{n,1}$ but with the induced metric coming from the Minkowski ($\pm$)-unit sphere through the radial projection and two fold covering of $\overline{K}^n$.

Note that our metric is negative of the original Lorentzian metric and the norm square of a tangent vector is positive. Hence this computation suggests us to choose a negative sign for a natural choice of sign for the norm of a tangent vector in Lorentzian part of $\mathbb{S}^1_H$, and for tangent vectors in the radial direction in Lorentzian part of general $\mathbb{S}^n_H$ as well. Similarly it is not hard to check the clockwise contour integral of the volume form gives sign $-i^{n-1}$ for Lorentzian part. If $n=2$, the sign for the volume is $-i$ and this suggests us that i is the natural choice for the sign for the norm of the spacelike tangent vector in the Lorentzian part of $\mathbb{S}^2_H$ and in $\mathbb{S}^n_H$ as well. This gives us a natural choice of signs for the norm of various tangent vectors which are compatible with volume form on $\mathbb{S}^n_H$. We summarize as follows.

A tangent vector on the hyperbolic part on $\mathbb{S}^n_H$ has a positive real norm, and a tangent vector on the Lorentzian part on $\mathbb{S}^n_H$ has a negative real, zero, or positive pure imaginary norm depending on whether it is timelike, null, or spacelike respectively.

We can see various similarities between the Euclidean sphere $\mathbb{S}^n$ and the hyperbolic sphere $\mathbb{S}^n_H$ in the following listed theorems (see [1], [3]).

Theorem 1. *vol($\mathbb{S}^n_H$)= i^n·vol($\mathbb{S}^n$).*

Remark. If we change the d as $1+\epsilon i$, we have different relation between vol($\mathbb{S}^n_H$) and vol($\mathbb{S}^n$): vol($\mathbb{S}^n_H$)= $(-i)^n$ vol($\mathbb{S}^n$).

Theorem 2. *The total length of any great circle is $2\pi i$.*

Theorem 3. *The k-dimensional volume of any k-dimensional geodesic sphere is vol ($\mathbb{S}^k_H$).*

Theorem 4. *For any vectors x and y in $\mathbb{S}^n_H$ and a point p in $x^\perp \cap y^\perp$, we have*

$$\langle x,y\rangle = \|x\|\|y\|\cos\angle(x_p,y_p)$$
$$\langle x,y\rangle = \|x\|\|y\|\cosh d_H(x,y).$$

Theorem 5. *The area of a triangle in $\mathbb{S}^2_H$ is $\pi - A - B - C$, where A,B,C are the angles of the given triangle.*

Theorem 6. *Letting A,B,C stand for the angles and a,b,c for the extended hyperbolic length of opposite sides, we obtain the hyperbolic cosine law and the dual cosine law on the hyperbolic sphere $\mathbb{S}^2_H$,*

$$\cos C = \frac{\cosh a\cosh b - \cosh c}{\sinh a \sinh b}$$
$$\cosh c = \frac{\cos A\cos B + \cos C}{\sin A\sin B}.$$

Also we have the spherical cosine law and dual cosine law on the spherical sphere $\mathbb{S}^2_S$, where a,b,c represent the extended spherical length,

$$\cos C = \frac{\cos c - \cosh a\cosh b}{\sin a \sin b}$$
$$\cos c = \frac{\cos A\cos B + \cos C}{\sin A\sin B}.$$

Theorem 7. *Letting A,B,C stand for the angles and a,b,c for the extended hyperbolic length of opposite sides, we obtain the hyperbolic sine law on the hyperbolic sphere $\mathbb{S}^2_H$,*

$$\frac{\sinh a}{\sin A} = \frac{\sinh b}{\sin B} = \frac{\sinh c}{\sin C}.$$

Also we have the spherical sine law on the spherical sphere $\mathbb{S}^2_S$, where a,b,c represent the extended spherical length,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

Remark. The spherical sphere $\mathbb{S}^n_S$ is made by the following ϵ-metric on $\mathbb{R}^n$:

$$ds^2_\epsilon = -\left(\frac{\Sigma x_i dx_i}{d^2_\epsilon - |x|^2}\right)^2 - \frac{\Sigma dx_i^2}{d^2_\epsilon - |x|^2}.$$

References

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