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DETERMINANT OF A DIRAC OPERATOR ON A COMPACT MANIFOLD WITH BOUNDARY

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ABSTRACT. In this note we discuss the relation between well-posed boundary conditions of a Dirac operator defining the regularized determinant and the asymptotic expansions of the trace of the heat kernel. We also discuss the zeta functions and eta functions associated to well-posed boundary conditions

1. INTRODUCTION

The regularized determinant of an elliptic operator is a spectral invariant, which is used importantly in geometry, topology anf physics. Specially the regularized determinant of the Laplacian is used by Ray and Singer in [RS] to define an analytic torsion, which is an analytic counterpart of Reidemeister torsion. There is also an important problem in quantum field theory to give a correct definition of the determinant of a Dirac operator on a compact manifold with boundary. On a compact closed manifold, the regularized determinant of a Dirac operator is well-defined by the usual method of zeta function from the work of Seeley in [S1]. Indeed, if D is a Dirac operator on a compact closed manifold M, the spectrum of D is distributed in a discrete way from $-\infty$ to ∞. Now let $\{\lambda_k\}$ and $\{\mu_k\}$ be all positive and negative eigenvalues of D, where λ_k, $\mu_k \in \mathbb{R}^+$. Set

$$\eta_D(s) = \sum_k \lambda_k^{-s} - \sum_k \mu_k^{-s}, \zeta_D(s) = \sum_k \lambda_k^{-s} + \sum_k (-\mu_k)^{-s},$$

$$\zeta_{D^2}(s) = \sum_k \lambda_k^{-2s} + \sum_k \mu_k^{-2s},$$

where all these functions are holomorphic for $Res > dimM$.

Then

$$\begin{aligned}\zeta_D(s) &= \sum_k (\frac{\lambda_k^{-s} + \mu_k^{-s}}{2} + \frac{\lambda_k^{-s} - \mu_k^{-s}}{2}) + (-1)^{-s} \sum_k (\frac{\lambda_k^{-s} + \mu_k^{-s}}{2} - \frac{\lambda_k^{-s} - \mu_k^{-s}}{2}) \\ &= \frac{1}{2}(\zeta_{D^2}(\frac{s}{2}) + \eta_D(s)) + \frac{1}{2}e^{-i\pi s}(\zeta_{D^2}(\frac{s}{2}) - \eta_D(s))\end{aligned}$$

For $Res > dimM$,

$$\begin{aligned}\zeta'_D(s) = \frac{1}{2}(\frac{1}{2}\zeta'_{D^2}(\frac{s}{2}) + \eta'_D(s)) - \frac{i\pi}{2}e^{-i\pi s}(\zeta_{D^2}(\frac{s}{2}) - \eta_D(s)) + \\ \frac{1}{2}e^{-i\pi s}(\frac{1}{2}\zeta'_{D^2}(\frac{s}{2}) - \eta'_D(s)).\end{aligned}$$

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After taking analytic continuation,

$$\zeta_D'(0) = \frac{1}{2}\zeta_{D^2}'(0) - \frac{i\pi}{2}(\zeta_{D^2}(0) - \eta_D(0)).$$

Hence,

$$Det_\zeta D = e^{-\zeta_D'(0)} = e^{\frac{\pi i}{2}(\zeta_{D^2}(0) - \eta_D(0))} \cdot e^{-\frac{1}{2}\zeta_{D^2}'(0)}.$$

If D is a Dirac operator defined on a compact manifold M with boundary and B is an appropriate boundary condition, then we consider the realization operator D_B, which acts like D but the domain is restricted to $\{f \in Dom(D) | Bf = 0\}$. Then along the same line as the closed case, we define (*i.e.* [SW])

$$Det_\zeta D_B = e^{\frac{\pi i}{2}(\zeta_{D_B^2}(0) - \eta_{D_B}(0))} \cdot e^{-\frac{1}{2}\zeta_{D_B^2}'(0)}.$$

¿From the definition we see that $Det_\zeta D_B$ is defined when $\zeta_{D^2}(s)$ and $\eta_D(s)$ have regular values at $s = 0$.

In this note, we give some examples of well-posed boundary conditions and discuss the general condition for B to satisfy the above properties.

2. Dirac operators and well-posed boundary conditions

Let M be a compact manifold of dimension n with boundary ∂M and $E \to M$ be a bundle of Clifford module. Near ∂M, we choose a collar neighborhood U so that U is diffeomorphic to $\partial M \times [0, r)$ for some $r \in \mathbb{R}^+$ and $E|_U$ is diffeomorphic to $E|_{\partial M} \times [0, r)$. We also choose a Riemannian metric g so that $g|_U$ is a product metric. By choosing a connection ∇ on E which is compatible to the Clifford structure on E, we can define a Dirac operator D acting on $C^\infty(M, E)$ as $D = \sum_i e_i \cdot \nabla_{e_i}$, where '$\cdot$' is a Clifford multiplication and $\{e_i\}_{i=1}^n$ is an orthonormal basis for $T_x M$ for $x \in M$. Then in a collar neighborhood U it is a well-known fact that D can be expressed as follows. $D = \sigma(\partial_{x_n} + A)$, where x_n is a normal coordinate to ∂M, A is a self-adjoint first order elliptic differential operator acting on $C^\infty(\partial M, E|_{\partial M})$, σ is a bundle automorphism and A, σ do not depend on the normal coordinate x_n. In fact, σ is given by Clifford multiplication of inward unit normal vector. Hence

$$\sigma^2 = -Id, \sigma^* = -\sigma, \sigma A = -A\sigma.$$

Now we define the Cauchy data space as

$$N^{(s-\frac{1}{2})} = \{u|_{\partial M} | u \in H^s(E), Du = 0\}.$$

Then we define the projection C called Calderon projector as

$$C : H^{s-\frac{1}{2}}(E|_{\partial M}) \to N^{(s-\frac{1}{2})}.$$

In fact, it is known by Calderon and Seeley (*i.e.*[S2], [BW]) that C is a pseudodifferential operator of order 0 and the principal symbol $\sigma_L(C)$ is the projection of the positive eigenspace of $\sigma_L(A) : T^*(\partial M) - 0 \to End(E|_{\partial M})$.

Definition Let $B : C^\infty(E|_{\partial M}) \to C^\infty(E|_{\partial M})$ be a ΨOD of order 0. B is called a well-posed boundary condition if the following two conditions are satisfied.
(i) For any $r \in \mathbb{R}$, $B : H^s(E|_{\partial M}) \to H^s(E|_{\partial M})$ has a closed range.

(ii) For each $(x', \xi') \in T^*(\partial M)$ with $|\xi'| = 1$, the principal symbol $\sigma_L(B)(x', \xi')$ maps the range of $\sigma_L(C)(x', \xi')$ isomorphically onto the range of $\sigma_L(B)(x', \xi')$.

For $s \geq 1$, we define the realization D_B as follows.

$$D_B : \{u | u \in H^s(E), Bu = 0\} \to H^{s-1}(E) \text{ by } D_B(u) = D(u).$$

Then from Seeley (*i.e.* [S2]), D_B is a Fredholm operator and has a discrete spectrum with $(D_B)^* = D_{(I-B^*)\sigma^*}$.

We now give some basic examples of well-posed boundary condition.

(1) $B = C$ is well-posed.

(2) If B is well-posed and P is an invertible operator on ∂M, then PB is well-posed. (In fact, PB is an equivalent boundary condition as B.)

(3) Let $\Pi_{\geq}$, $\Pi_{<}$ be the projections onto non-negative and negative eigenspaces of A, respectively. It is known by Scott and Grubb (*i.e.*[G]) that $C - \Pi_{\geq}$ is a smoothing operator, *i.e.* the total symbols of C and $\Pi_{\geq}$ are same. Hence $\Pi_{\geq}$ and $\Pi_{<} = \sigma\Pi_{\geq} - pr_{KerA}$ are well-posed.

(4) Any orthogonal pseudodifferential projector having the principal symbol $\sigma_L(C)(x', \xi')$ is well-posed.

Now we set $Gr^*(D)$ the set of all orthogonal pseododifferential projcetors satisfying the following conditions.

$$B^2 = B, B^* = B, \sigma_L(B) = \sigma_L(\Pi_{\geq}), \sigma B + B\sigma = \sigma.$$

We also set

$$Gr^*_\infty(D) = \{B \in Gr^*(D) | B - \Pi_{\geq} \text{ is a smoothing operator }\}.$$

Then for any $B \in Gr^*$, B is a well-posed boundary condition and $(D_B)^* = D_{(I-B^*)\sigma^*} = D_{-(I-B)\sigma} = D_{-\sigma B} = D_B$. Hence, D_B is a self-adjoint operator.

Remark (1) $\Pi_{\geq} \in Gr^*_\infty(D)$ if and only if $KerA = 0$.

(2) If $KerA \neq 0$, set $L = \{\phi \in KerA | \sigma\phi = \phi\}$ so that $KerA = L \oplus L^\perp$ and $B = \Pi_{\geq} + \Pi_L$. Then $B \in Gr^*_\infty(D)$.

(3) C is not generally orthogonal (*i.e. not self-adjoint*). Let $C^\perp = CC^*(CC^* + (I-C)^*)(I-C))^{-1}$. then $ImC^\perp = ImC$, $(C^\perp)^* = C^\perp$ and $\sigma_L(C^\perp) = \sigma_L(C)$. Hence $C^\perp \in Gr^*(D)$.

The following results are due to Wojciechowski (see [W]).

(1) For any $B \in Gr^*_\infty(D)$, $\eta_{D_B}(s)$ and $\zeta_{(D_B)^2}(s)$ have regular values at $s = 0$ and so the regularized determinant of D_B is well-defined.

(2) For any B_1, $B_2 \in Gr^*_\infty(D)$, $\zeta_{(D_{B_1})^2}(0) = \zeta_{(D_{B_2})^2}(0)$.

3. Asymptotics of the traces of heat kernels of D_B and D_B^2

We start from defining the zeta function and the eta function associated to the operators D_B and D_B^2 with a given well-posed boundary condition B. Let $\{\lambda_i\}_{i=1}^\infty$ be all nonzero eigenvalues of D_B. Define $\eta_{D_B}(s) = \sum_{\lambda_i} sign(\lambda_i)|\lambda_i|^{-s}$ and $\zeta_{D_B^2}(s) = \sum_{\lambda_i} |\lambda_i|^{-2s}$. Then $\eta_{D_B}(s)$ is holomorphic for $Res > dimM$ and $\zeta_{D_B^2}(s)$ is holomorphic for $Res > \frac{dimM}{2}$. Note that for $\lambda > 0$,

$$\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\lambda t} dt = \frac{1}{\Gamma(s)} \int_0^\infty (\frac{u}{\lambda})^{s-1} e^{-u} \frac{1}{\lambda} dt = \lambda^{-s}.$$

Hence, for $Res > \frac{dim M}{2}$

$$\zeta_{D_B^2}(s) = \sum_{\lambda_i} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-|\lambda_i|^2 t} dt = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} Tr(e^{-tD_B^2} - pr_{Ker D_B}) dt$$
$$= \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} Tr(e^{-tD_B^2} - pr_{Ker D_B}) dt$$
$$+ \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} Tr(e^{-tD_B^2} - pr_{Ker D_B}) dt,$$

where $\frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} Tr(e^{-tD_B^2} - pr_{Ker D_B}) dt$ is a holomorphic function for whole $s \in \mathbb{C}$. As the same way, for any $\lambda \in \mathbb{R} - \{0\}$

$$\frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{\frac{s-1}{2}} \lambda e^{-t\lambda^2} dt = \begin{cases} \lambda^{-s} & \text{if } \lambda > 0, \\ -|\lambda|^{-s} & \text{if } \lambda < 0. \end{cases}$$

Hence,

$$\eta_{D_B}(s) = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{\frac{s-1}{2}} Tr(D_B e^{-tD_B^2}) dt$$
$$= \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^1 t^{\frac{s-1}{2}} Tr(D_B e^{-tD_B^2}) dt + \frac{1}{\Gamma(\frac{s+1}{2})} \int_1^\infty t^{\frac{s-1}{2}} Tr(D_B e^{-tD_B^2}) dt,$$

where $\frac{1}{\Gamma(\frac{s+1}{2})} \int_1^\infty t^{\frac{s-1}{2}} Tr(D_B e^{-tD_B^2}) dt$ is also a holomorphic function for whole $s \in \mathbb{C}$. Hence we are interested in the asymptotic expansion of ${D_B}^l e^{-tD_B}$ as $t \to 0$ for $l = 0$ or 1. The following asymptotic series are proved by Seeley, Grubb, Bruning and Lesch (see [BL], [G], [GS]).

As $t \to 0$,

$$Tr(e^{-tD_B^2}) \sim a_0 t^{-\frac{n}{2}} + \sum_{j=0}^\infty (a_j + b_j) t^{\frac{j-n}{2}} + \sum_{j=0}^\infty (c_j log t + d_j) t^{\frac{j}{2}},$$

$$Tr(D_B e^{-tD_B^2}) \sim \tilde{a}_0 t^{-\frac{n}{2}} + \sum_{j=0}^\infty (\tilde{a}_j + \tilde{b}_j) t^{\frac{j-n-1}{2}} + \sum_{j=0}^\infty (\tilde{c}_j log t + \tilde{d}_j) t^{\frac{j-1}{2}}.$$

Here a_j, $\tilde{a}_j$ are expressed by integrals of symbols of D on M, b_j, c_j, $\tilde{b}_j$, $\tilde{c}_j$ are expressed by the integrals of symbols of D and B on ∂M, and d_j, $\tilde{d}_j$ are not computed by the integrals of local densities, *i.e.* d_j, $\tilde{d}_j$ are determined by global data.

Note that

(1) $\frac{1}{\Gamma(s)} \int_0^1 t^{s-1} t^{\frac{j-n}{2}} dt = \frac{1}{\Gamma(s)} \frac{1}{s + \frac{j-n}{2}}$.

(2) $\frac{1}{\Gamma(s)} \int_0^1 t^{s-1} t^{\frac{j}{2}} log t dt = \frac{1}{\Gamma(s)} \frac{-1}{(s + \frac{j}{2})^2}$.

(3) $\frac{1}{\Gamma(\frac{s+1}{2})} \int_0^1 t^{\frac{s-1}{2}} t^{\frac{j-n-1}{2}} dt = \frac{1}{\Gamma(\frac{s+1}{2})} \frac{2}{s+j-n}$.

(4) $\frac{1}{\Gamma(\frac{s+1}{2})}\int_0^1 t^{\frac{s-1}{2}} t^{\frac{j-1}{2}} logtdt = \frac{1}{\Gamma(\frac{s+1}{2})}\frac{-4}{(s+j)^2}$.

Hence, at $s=0$,
(1) $\zeta_{D_B{}^2}(s)$ has a simple pole and ite principal part is $\frac{-c_0}{s}$,
(2) $\eta_{D_B}(s)$ has a double pole and its principal part is $\frac{\tilde{a}_n+\tilde{b}_n+\tilde{d}_0}{\Gamma(\frac{n+1}{2})}\frac{2}{s} + \frac{-4}{\Gamma(\frac{1}{2})}\frac{\tilde{c}_0}{s^2}$.

In fact, the coefficients of $logt$ and $t^{\frac{-1}{2}}$, $t^{\frac{-1}{2}} logt$ give the principal parts of $\zeta_{D_B{}^2}(s)$ $\eta_{D_B}(s)$, respectively.

Hence, we get a conclusion that the regularized determinant of D_B is well-defined if and only if $c_0 = 0$, $\tilde{a}_n+\tilde{b}_n+\tilde{d}_0 = 0$ and $\tilde{c}_0 = 0$ in the above asymptotic expansions.

Remark Assume that $\zeta_{D_B{}^2}(s)$ and $\eta_{D_B}(s)$ have regular values at $s=0$. Then
(1) $\zeta_{D_B{}^2}(0) = a_n + b_n + d_0 - dimKerD_B$ *i.e.* $\zeta_{D_B{}^2}(0) - dimKerD_B$ is not lacally determined but in a closed manifold it is locally determined with the value a_n
(2) $\eta_{D_B}(0)$ is not locally determined, either. $\eta_D(0)$ is not lacally determined even in a closed manifold.

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