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THE LOCALIZATION THEOREM

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ABSTRACT. We survey the localization theorem, following Atiyah and Bott.

1. INTRODUCTION

In this short note, we would like to introduce the localization theorem proven by Atiyah and Bott ([1]). The result dates back to the works of P.A. Smith and A. Borel. This well-known theorem have many applications. Recently, it was applied to enumerative geometry. Spaces concerned there are orbifolds. However, localization theorem is formally proven only for manifolds. Let X be a compact oriented smooth manifold and let T be a torus, i.e., a product of copies of the circle group. Suppose T acts on X smoothly. Denote by X^T the set of fixed points of X. It is always a submanifold, not necessary connected, according to the equivariant slice theorem ([4]). Let x be any element of X and let T_x be the isotropy subgroup of x. The equivariant slice theorem says that there is an invariant tubular neighborhood of the orbit $O(x)$ of x such that the neighborhood is equivariant diffeomorphic to $T \times_{T_x} N_x$ where N_x is the normal space to the orbit $O(x)$ at x. Notice that N_x has the induced T_x-action and here T acts on $T \times_{T_x} N_x$ as $t \cdot (t', n) = (tt', n)$. In this equivariant diffeomorphism it is required that the zero section of $T \times_{T_x} N_x$ corresponds to the orbit $O(x)$. The theorem is a result of the existences of an invariant metric and the exponential map of the metric. The localization theorem tells us that in a certain moderate case cohomology classes of X are determined by the restrictions of the classes to X^T.

Let T act on $X \times ET$ as $t \cdot (x, e) = (tx, et^{-1})$ for $t \in T$, $(x, e) \in X \times ET$ where ET is a total space of a universal principal T-bundle. In fact, ET is nothing but a contractible space with a free right T-action. Denote by X_T the quotient $X \times ET/T = X \times_T ET$ and call X_T the homotopic quotient of X. The reason is that when T acts on X freely, it is homotopic equivalent to the genuine quotient X/T. In general, they are homotopically different types. By means of the projection map from $X \times_T ET$ to BT the cohomology group $H^*(X_T)$ can be considered as a $H^*(BT, \mathbb{Q})$-module. $H^*(X_T, \mathbb{Q})$ is denoted by $H_T^*(X)$ and is called the equivariant cohomology of X. We say that a class α in $H^*(X_T)$ is a torsion element if $a\alpha = 0$ for some nonzero a in $H^*(BT)$. Let i denote the embedding map of X^T into X, then the localization theorem reads that $i^* : H_T^*(X) \to H_T^*(X^T)$ and $i_* : H_T^*(X^T) \to H_T^*(X)$ are isomorphisms modulo torsion modules, i.e., their kernels and cokernels are torsion modules. We will explain notation, a proof of the theorem following Atiyah and Bott. For an application, we will obtain an integration formula.

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2. Pushforwards

We will consider cohomology groups with coefficients in $\mathbb{Q}$ unless they are specified. Let X and Y be smooth compact oriented manifolds. Let $\dim Y - \dim X = k$ and let $f : X \to Y$ be a smooth map. Define the pushforward $f_* : H^*(X) \to H^{*+k}(Y)$ by the required property $(f_*(\alpha), \beta)_X = (\alpha, f^*\beta)_Y$ where $(.,.)_X$ and $(.,.)_Y$ are the Poincaré pairings in X and in Y, respectively. Geometrically, one would consider it as the following procedure. Take a Poincaré dual cycle C of α and then define the pushforward of α as the Poincaré dual of cycle $f(C)$.

Pushforwards satisfy the followings.

1. When f is a fibering, it is integration along the fiber as defined in ([4]). In particular, if Z be a submanifold of Y and i_Z, $i_{f^{-1}(Y)}$ are the inclusions of Z and $f^{-1}(Y)$, then
$$(f|_{f^{-1}(Y)})_* \circ i^*_{f^{-1}(Y)} = i^*_Z \circ f_*.$$
2. When f is an inclusion, it is factored as
$$f_* = j^* \circ \Phi_X.$$
Here j is the restriction map from $H^*(Y, Y - X)$ to $H^*(Y)$ and Φ is the Thom isomorphism from $H^*(X) \cong H^*(N_{X/Y})$ to $H^{*+n}(N_{X/Y}, N_{X/Y} - X) \cong H^*(Y, Y - X)$ where $N_{X/Y}$ is the normal bundle to X in Y.
3. (The projection formula)
$$f_*(\alpha f^* \beta) = f_*(\alpha)\beta.$$
The proof of it is purely algebraic and follows from the definition.

Let X and Y be a CW complex such that their n-skeletons X_n and Y_n are smooth compact oriented manifolds for all n. Notice that if $n > m$,
$$H_m(X, \mathbb{Z}) = H_m(X_n, \mathbb{Z})$$
and
$$H^m(X, \mathbb{Q}) = \mathrm{Hom}(H_m(X, \mathbb{Q}), \mathbb{Q}) = H^m(X_n, \mathbb{Q})$$
See theorem A.1 in Appendix A of ([5]) for a reference. Suppose f is a map from X to Y, which induces smooth maps f_n from X_n to Y_n for all n by means of restrictions. We define
$$f_* := \lim_{\leftarrow} (f_n)_*.$$
We use this method to define pushforwards between homotopic quotients.

3. Equivariant cohomology

Let T be a torus, a product of n many copies of the circle group. Let $ET \to BT$ be the universal T-principal bundle. BT is called the classifying space. They are unique upto homotopy equivalence. Take $BT = (\mathbb{C}P^\infty)^n$ and $ET = (S^\infty)^n$. See ([3]) for details. Notice that $(S^{2m+1})^n \to (\mathbb{C}P^m)^n$ approximates the bundle. Let X be a smooth manifold with a smooth action of T. Define the homotopic quotient X_T as $X \times_T ET$. The equivariant cohomology $H^*_T(X)$ is by definition the cohomology $H^*(X_T, \mathbb{Q})$ of the homotopic quotient. Using the natural quotient map from X_T to BT, we consider the cohomology as $H^*(BT)$-module. Let Y be a T-space and f be an equivariant map from X to Y. Then we define the pushforward f_* of $X_T \to Y_T$ using the approximations.

4. Localization Theorem

Consider

$$H^*_T(X) := \bigoplus_{n=0}^{\infty} H^n_T(X)$$

as $H^*(BT) := \bigoplus_{n=0}^{\infty} H^n(BT)$-module, using the pullback π^* of the projection $\pi : X_T \to BT$.

There is an invariant tubular neighborhood which is equivariant diffeomorphic to the equivariant normal bundle of X^T. Cover the rest by a finite number of invariant open subsets of form $T \times_{T_x} N_x$ for some non-fixed points x. Since $(T \times_{T_x} N_x)_T = (N_x)_{T_x}$, the module structure of $H^*_T(T \times_{T_x} N_x)$ induced from the $H^*(BT_x)$-module structure associated to $H^*(BT_x) \to H^*(BT)$. Since T_x is a proper subgroup of T, $H^*_T(T \times_{T_x} N_x)$ is a torsion module. In fact, it follows from the general fact.

Lemma: *If there is an equivariant map $X \to T/K$ where K is a proper Lie subgroup of T, then $\mathrm{Ann} H^*_T(X)$ is nontrivial.*

Proof: $H^*(BT_x) \leftarrow H^*(BT)$ is a projection with a nontrivial kernel.

Claim: Let R be an integral domain. If there is an exact sequence of R-modules $D \to E \to F$ such that D and F are torsion modules, then E is also a torsion module.

The proof of the claim is easy.

Proposition: *Let X be a compact manifold on which a torus T acts. Then $H^*_T(X - X^T)$ and $H^*(X - U, \partial(X - U))$ are torsion modules, where U is an invariant open tubular neighborhood of X^T.*

Proof: Cover $X - X^T$ by a finite number orbit neighborhoods U_i and U. Then by the lemma $H^*_T(U_i)$ are torsion modules and also by the lemma $H^*_T((\cup_{i=1}^k U_i) \cap U_{k+1})$ are torsion modules. Thus $H^*_T(X - U)$ is a torsion module which can be seen if one applies Mayer-Vietoris sequences inductively. In the step, we use the claim. Make U is smaller if necessary so that $X - U$ is T-equivariant isotropic to $X - X^T$. Now we conclude the first claim of the proposition. Similarly, covering $\partial(X - U)$ by orbit neighborhoods in $\partial(X - U)$, we see that $H^*_T(\partial(X - U))$ is a torsion module. Using a relative cohomology exact sequence, we conclude that $H^*(X - U, \partial(X - U))$ is a torsion module.

Theorem: *The kernel and cokernel of $i^* : H^*_T(X) \to H^*_T(X^T)$ are torsion modules.*

Proof: Consider an exact sequence $0 \to \Omega^*(X, X^T) \to \Omega^*(X) \to \Omega^*(X^T) \to 0$. Notice that $H^*_T(X, X^T)$ is a torsion module: It is isomorphic to $H^*_T(X - U, \partial(X - U))$ since U is equivariantly retractible to X^T.

Theorem: *The kernel and cokernel of $i_* : H^*_T(X^T) \to H^*_T(X)$ are torsion modules.*

Proof: Consider an exact sequence $0 \to \Omega^*_T(X, X - X^T) \to \Omega^*_T(X) \to \Omega^*_T(X - X^T) \to 0$. Since $H^*_T(X, X - X^T)$ is isomorphic to $H^*_T(X^T)$ under the Thom isomorphism and the pushforwards factored through Thom isomorphisms, the proof follows.

5. Integration Formula

Lemma: *For* $\alpha \in H_T^*(X^T)$,

$$i^* i_*(\alpha) = \alpha Euler_T(N_{X^T}).$$

Proof: In each finite dimensional approximation, the formula is easily seen to be true using the geometric definition of the pushforward. So, the lemma holds.

With this formula and the localization theorem, we will obtain an integration formula. Let H_T^* denote $H^*(BT)$ and $H_{(T)}^*$ denote the fractional field of an integral domain $H^*(BT)$.

Theorem: *For any* $\gamma \in H_T^*(X) \otimes_{H_T^*} H_{(T)}^*$,

$$\pi_* \gamma = \sum_F \pi_*^F \frac{i_F^* \gamma}{Euler_T(N_{F/X})},$$

where F *are connected components of* X^T *and* π^F *are the projections from* $F_T = F \times BT \to BT$.

The proof of the integration formula is *purely algebraic.* First we need to show that $Euler(N_{X^T})$ is invertible in $H_T^*(X) \otimes H_{(T)}^*$. It follows from the above formula for $i^* i_*$ and the consideration of 1 as the image of some element under the map $i^* i_*$.

Secondly, $\gamma = i_* \frac{i^* \gamma}{Euler_T(N_{X^T/X})}$ since γ is the image of some element under the map i_*. Now the proof follows.

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