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A TOUR ON CLASSICAL THEOREMS ON MULTISECANTS OF PROJECTIVE VARIETIES AND PROBLEMS

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Abstract. This note is intended to introduce some interesting classical results on multisecant lines of projective varieties, varieties swept out by multisecants and open questions regarding them to mathematicians with modest background on algebraic geometry. In addition, such questions have been solved and understood by using modern projective techniques on vector bundles and by new results around them.

§0 Generalities on Multisecants of Projective Varieties.

Let X be a nondegenerate smooth projective subvariety of degree d, dimension n and codimension e and let $S_k(X)$ be the closure of all k-secant lines of X in $\mathbb{P}^N$ defined over the complex number field $\mathbb{C}$. We can consider a descending filtration associated to X:

$$\mathbb{P}^N \supseteq S_2(X) \supseteq S_3(X) \supseteq \cdots \supseteq S_{t-1}(X) \supseteq S_t(X) = \cdots = S_\infty(X),$$

where $S_\infty(X)$ is the subvariety swept out by the lines contained in the variety X. It is clear that the first number t for which $S_t(X) = S_\infty(X)$ satisfies the inequality $t \leq d-e+2$ by the generalized Bezout theorem. Note that $S_2(X)$ is also irreducible for an irreducible variety X. In particular, if X is contained in a hypersurface F of degree m, then $S_k(X) \subseteq F$ for all $k > m$.

Similarly, let $\Sigma_k(X)$ be the locus of all k-secant lines of X in the Grassmannian $\mathbb{G}(1,N)$. We have also a descending filtration associated to X:

$$\mathbb{G}(1,N) \supseteq \Sigma_2(X) \supseteq \Sigma_3(X) \supseteq \cdots \supseteq \Sigma_{t-1}(X) \supseteq \Sigma_t(X) = \cdots = \Sigma_\infty(X),$$

where $\Sigma_\infty(X)$ is the subvariety consisting of all the lines contained in the variety X. It is well known due to Z. Ran that $S_{n+2}(X)$ is at most $(n+1)$- dimensional, see the "(dimension+2)-secant lemma" in [R3]. It is quite useful to know the following result due to B. Segre:

Theorem(B. Segre, [BSe]**).** *Let $X^n \subset \mathbb{P}^N$ be an irreducible variety of dimension n. Let $\Sigma \subset \mathbb{G}(1,N)$ be a component of maximal dimension of the variety of lines contained in X. Then,*

(a) *if $\dim(\Sigma) = 2n-2$, then $X = \mathbb{P}^n$.*
(b) *if $\dim(\Sigma) = 2n-1$, then X is either a quadric or a scroll in $\mathbb{P}^2$'s over a curve.*

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One more remark is that projecting a smooth n-dimensional nondegenerate subvariety X of $\mathbb{P}^{2n+1}$ from a general point, we obtain a finite number $\delta(X)$ of "nodes" which is equal to the number of secant lines to X passing through a general point of $\mathbb{P}^{2n+1}$.

¿From now on, let us proceed with the dimension of X in order to explain multisecants and results about them.

§1 Multisecants of Curves

For a nondegenerate smooth curve C of degree d in $\mathbb{P}^3$, $\dim\Sigma_2(C) = 2$ and $S_2(C) = \mathbb{P}^3$. So, such a curve has necessary apparent nodes whose number is $\delta(C) = \frac{(d-1)(d-2)}{2} - g(C) > 0$. It is easy to see that the only smooth curve in $\mathbb{P}^3$ having one apparent node is the rational normal curve of degree 3. The following "generalized trisecant lemma" has many applications in classical projective algebraic geometry. In particular it is the special case of the "general position lemma" for an arbitrary curve in $\mathbb{P}^N$.

Proposition 1.1. *Let C be a nondegenerate reduced irreducible curve of degree d in $\mathbb{P}^3$. Let $\Sigma_3(C)$ be the locus of all trisecant lines of C in the grassmannian $\mathbb{G}(1,3)$. Then,* $\dim \Sigma_3(C) = 1$ *unless C is either a twisted cubic curve or an elliptic normal quartic curve.*

Proof. First note that $\dim \Sigma_3(C) \leq 1$. This is actually caused by the classical "trisecant lemma " for curves in $\mathbb{P}^3$, see [ACGH, pp 109-111] for a proof. If C has no trisecants, then C should be smooth and the projection of C from a general point of C is an isomorphism onto a smooth plane curve C' of degree $d-1$. In particular, the genus of C is given by and bounded by Castelnuovo as follows:

$$g(C) = \frac{(d-2)(d-3)}{2} \leq \begin{cases} \dfrac{d^2}{4} - d + 1 & \text{if d is even.} \\ \dfrac{d^2-1}{4} - d + 1 & \text{if d is odd.} \end{cases}$$

So, we have $2 \leq d \leq 4$. If d=3, it is the twisted cubic and if $d = 4$, then C is either an elliptic normal curve of two quadrics or a rational quartic curve of type(1,3) lying on a smooth quadric surface. However, a curve of type (1,3) has one of rulings of the quadric as trisecants. So we are done. □

Remark 1.2. *The proof of Proposition 1.1 shows in fact that a space curve has either infinitely many trisecants or none!*

Question 1.3. *Consider the trisecant surface $S_3(C)$ of a space curve $C \subset \mathbb{P}^3$ and describe simple geometric properties of it. What can be said about its singularities?*

It is closely related to the following numbers $n(C)$ and $m(C)$, which are defined respectively as the biggest number such that C has an infinite number of n-secant lines and the number of $n(C)$-secant lines passing through a generic point of $S_3(C)$. For example, if C has finitely many 4-secant lines and tangential trisecant lines, then $C \subset \mathrm{Sing}(S_3(C))$, see [Be]. A classical formula of Berzolari gives the number of trisecant lines to C in $\mathbb{P}^3$ meeting a general line in $\mathbb{P}^3$:

$$\frac{(d-1)(d-2)(d-3)}{3} - (d-2)g \qquad (1.3.0)$$

In case, if no multiplicities are involved, Berzolari's formula computes the degree of the trisecant surface $S_3(C)$.

In general, one can expect that C has only finitely many higher order secants. Note the following Cayley formula on the number(with multiplicity) of quadrisecant lines to a smooth curve C of degree d and genus g in $\mathbb{P}^3$;

(1.3.1) $$\frac{(d-2)(d-3)^2(d-4)}{12} - \frac{(d^2-7d+13-g)g}{2},$$

where a quintisecant line counts in general as 5 quadrisecants and negative result implies that C has an infinite number of quadrisecants. For example, consider a curve of type(4,4) lying on a smooth quadric surface Q in $\mathbb{P}^3$. The Cayley formula yields a negative number -4 but both ruling of quadric Q are quadrisecant to C.

§2 Multisecants of Surfaces.

For any smooth irreducible surface X in $\mathbb{P}^4$, we get $S_2(X) = \mathbb{P}^4$ (a general fact!) and $S_4(X) \neq \mathbb{P}^4$ (due to Z. Ran, [R3]).

On the other hand, for trisecant lines of a smooth surface X, we have the following lemma which is a slight generalization of the trisecant lemma for a space curve in $\mathbb{P}^3$.

Lemma 2.1. *Let X be a smooth surface of degree d in $\mathbb{P}^4$. Let $\Sigma_3(X)$ be the locus of all trisecant lines of X in the Grassmannian $\mathbb{G}(1,4)$. Then,*

$$\dim \Sigma_3(X) = 3$$

unless X is either a rational cubic or a complete intersection of two quadrics.

Proof. This is actually caused by "generalized trisecant lemma" for curves in $\mathbb{P}^3$. For a smooth surface X in $\mathbb{P}^4$, consider the incidence correspondence

$$\begin{array}{ccc} \Phi_2 = \{(\ell, H) : \ell \subset H\} \subset \Sigma_3(X) \times \mathbb{P}^{4*} & \xrightarrow{\pi_2} & \mathbb{P}^{4*} \\ \downarrow{\scriptstyle \pi_1} & & \\ \Sigma_3(X) \subset \mathbb{G}(1,4) & & \end{array}$$

Then, the fibers of π_1 are all irreducible of dimension 2. So $\dim \Phi_2 = \dim \Sigma_3(X) + 2$. By the "generalized trisecant Lemma" for curves in $\mathbb{P}^3$, all generic fibers of π_2, ${\pi_2}^{-1}(H) = \{(\ell, H) : \ell \subset H\} = \Sigma_3(X \cap H)$ are exactly 1-dimensional because it is neither a twisted cubic curve nor a complete intersection of two quadrics, so $\dim \Phi_2 = 5$ and consequently, $\dim \Sigma_3(X) = 3$. □

By the way, since $\Sigma_3(X)$ is 3-dimensional or empty, we know that $S_3(X)$ is at most 4-dimensional and we can imagine that $S_3(X) = \mathbb{P}^4$ for almost all surfaces in $\mathbb{P}^4$. Therefore, the number of trisecant lines to X through a generic point in $\mathbb{P}^4$ is always finite and given by the formula, citeAu: let t be the number of trisecant lines to X through a generic point in $\mathbb{P}^4$. Then we have

(2.1.0) $$t = \binom{d-1}{3} - \pi(d-3) + 2\chi - 2$$

where χ is the Euler characteristic of X and π is the sectional genus of X.

In this direction, F. Severi showed that if $S_3(X) \neq \mathbb{P}^4$ (equivalently $t = 0$), then X is either contained in a quadric hypersurface (trivial case!) or $S_3(X)$ is a hypersurface ruled in planes, each plane intersecting the surface in a curve of degree at least 3, see Theorem 0.1 and [FSe].

It was also conjectured by C. Peskine and proved by A. Aure that the elliptic quintic scroll is the only *nontrivial* example such that $S_3(X) \neq \mathbb{P}^4$, $H^0(\mathcal{I}_{X/\mathbb{P}^4}(2)) = 0$ [Au]. Note that the elliptic quintic scroll X can be defined as the degeneracy locus of five generic sections s_1, s_2, s_3, s_4, s_5 of $H^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(3))$ with resolution

$$0 \to \mathcal{O}_{\mathbb{P}^4}^{\oplus 5} \xrightarrow{\varphi=(s_i, 1\leq i\leq 5)} \Omega^2_{\mathbb{P}^4}(3) \xrightarrow{(\bigwedge^5 \varphi)^*} \mathcal{I}_X(3) \to 0.$$

A. Aure's idea is to use formulas (2.1.0) and (2.2.0). That is, if $t = 0$ and X is not contained in a cubic hypersurface then the sectional genus π is bigger than the upper bound given by J. Harris. Therefore, if $t = 0$ then X should be contained in a cubic hypersurface and then he used formula (2.2.0) to pick up only one example, i.e. the elliptic quintic scroll.

Remark 2.2.

In general, we might guess that for a general surface $X \subset \mathbb{P}^4$

$$\dim \Sigma_{3+i}(X) = 3 - i.$$

However, this is not always true for all smooth surfaces in $\mathbb{P}^4$. In particular, the following numbers if it is finite can be computed, see [LB]: the number of 4-secants to X which meet a general line is

$$2\binom{d}{4} + t(d-3) + h - \delta\binom{d-3}{2} \qquad (2.2.0)$$

where $\delta = \binom{d-1}{2} - \pi$ is the degree of the double curve Γ of a general projection of X to $\mathbb{P}^3$ and h is the number of apparent double points on Γ.

Similarly, the number of 5-secants to X which meet a general plane and the number of 6-secants of X can be also obtained in terms of d, π, χ.

Question 2.3(Severi problem on a surface in $\mathbb{P}^5$). *Classify smooth surfaces in $\mathbb{P}^5$ having only one secant line through a general point of $\mathbb{P}^5$. Thus, their generic projection to $\mathbb{P}^4$ has just one node.*

In 1901 F. Severi proved that the smooth surfaces in $\mathbb{P}^5$ having one apparent node are only the rational normal scroll(s) of degree 4 or the Del Pezzo surface of degree 5. Unfortunately, his argument has a gap (This gap was remarked first by Ciliberto and Sernesi): he does not consider the case of surfaces containing infinitely many planar curves, for detail and further results, see [Ru].

§3 Multisecants of Threefolds.

For a smooth threefold X of codimension two, we have $S_2(X) = \mathbb{P}^5$ and $S_5(X) \neq \mathbb{P}^5$. As like in Lemma 2.1, we can show that

$$\dim \Sigma_3(X) = 5$$

unless X is either a cubic scroll or a complete intersection of two quadrics.

It is quite natural to ask whether *nontrivial* smooth threefolds in $\mathbb{P}^5$ such that $S_{k+1}(X) \neq \mathbb{P}^5, H^0(\mathcal{I}_X(k)) = 0$, for $k = 2, 3$ exist or not. This question is to extend the work of F. Severi [FSe] and A. Aure [Au] to the case of smooth threefolds in $\mathbb{P}^5$ without assuming the subcanonical condition on X.

Theorem 3.1. *Let X be a smooth threefold of degree d in $\mathbb{P}^5$.*

(a) $\dim S_3(X) \leq 4$ *if and only if X is contained in a hyperquadric;*

(b) *Suppose that X is not contained in a cubic hypersurface. If the following two multiplicative maps*

$$H^0(\mathcal{O}_{\mathbb{P}^5}(1)) \otimes H^1(\mathcal{O}_X(1)) \to H^1(\mathcal{O}_X(2))$$

$$H^0(\mathcal{O}_{\mathbb{P}^5}(1)) \otimes H^2(\mathcal{O}_X) \to H^1(\mathcal{O}_X(1))$$

are injective, then $S_4(X) = \mathbb{P}^5$.

Remark 3.2.

(a) *The main ideas of Theorem 3.1 is to use Le Potier vanishing theorem for ample vector bundles and the generalized Castelnuovo-Mumford lemma for globally generatedness of vector bundles. For detail, see* [K2].

(b) *it is conjectured due to Peskine-Zak that there are very few examples such that* $\dim H^0(\mathbb{P}^5, \mathcal{I}_{X/\mathbb{P}^5}(3)) = 0$, $\dim S_4(X) \leq 4$ *and either* $\dim H^1(X, \mathcal{O}_X(1)) \neq 0$ *or* $\dim H^2(X, \mathcal{O}_X) \neq 0$.

Question 3.3.

(a) *Classify all smooth threefolds in $\mathbb{P}^5$, not contained in a cubic hypersurface but $S_4(X) \neq \mathbb{P}^5$.*

(b) *Find out the formula counting the number of 4-secant lines to X passing through a general point of $\mathbb{P}^5$.*

(c) *For a smooth threefold X of any codimension, show that $S_4(X)$ is at most 4-dimensional under the mild assumption.*

Remark 3.4. *Let $X \subset \mathbb{P}^N$ be a smooth threefold of arbitrary codimension. We know that if $\dim S_4(X) \leq 4$ then the Castelnuovo-Mumford regularity conjecture turns out to be true,* [K1], [R2] *i.e.*

$$\operatorname{reg} X \leq \deg X - \operatorname{codim} X + 1.$$

In fact, Z. Ran claimed that for $N \geq 9$, $\dim S_4(X) \leq 4$ and as a result, regularity conjecture is true in this case, [R2]. *See also* [K1] *for Castelnuovo regularity of smooth threefolds and fourfolds.*

§4 Codimension Two in General.

We note that a smooth variety of codimension 2 in $\mathbb{P}^{n+2}$ is always subcanonical for $n \geq 4$ and consequently, by Serre's construction, such a variety can be defined as a zero locus of a section of a rank two vector bundle over $\mathbb{P}^{n+2}$.

Theorem 4.1(Z. Ran). *Let $X^n \subset \mathbb{P}^{n+2}, n \geq 2$ be a locally complete intersection projective variety of degree d and let $\mathcal{N}$ be the normal bundle of X such that $\bigwedge^2 \mathcal{N} \simeq \mathcal{O}_X(v), v \in \mathbb{Z}$. Then the followings are equivalent for $m \leq n$:*

(a) $S_{m+1}(X) \neq \mathbb{P}^{n+2}$

(b) *X is contained in a hypersurface F of degree m.*

Proof. His idea is to combine an enumerative geometry with many results about a vector bundle appearing in the Serre's construction, see [R1].

The same equivalence is also true for the category of arithmetically Cohen-Macaulay, not necessary smooth, projective subvarieties of codimension two, see [K2].

Proposition 4.2. *Let $X^n \subset \mathbb{P}^{n+2}, n \geq 2$ be an arithmetically Cohen-Macaulay, not necessary smooth, variety of codimension two. then for all $m \leq n$, we have*

$$\dim S_{m+1}(X) \leq n+1 \textit{ if and only if } H^0(\mathbb{P}^{n+2}, \mathcal{I}_X(m)) \neq 0$$

Problem 4.3. *As Fyodor L. Zak pointed out, in general, for a projective variety $X^n \subset \mathbb{P}^N$ of small codimension such that $S_{m+1}(X) \neq \mathbb{P}^N$ and $H^0(\mathcal{I}_X(m)) = 0$, what can be said about X geometrically and cohomologically?*

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