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ON WEIERSTRASS POINTS

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ABSTRACT. The main goal of this survey note is to introduce Weierstrass points and gap sequences which is one of the classical and interesting object of curve theory. Also we will introduce the recent progress on related topics.

1. INTRODUCTION

One of a very classical topic in the study of an algebraic geometry is the study of Weierstrass points of nonsingular complex projective curve.

Let C be a nonsingular complex projective curve of genus $g \geq 2$ and $\mathcal{M}(C)$ denote the field of meromorphic functions on C and $\mathbb{N}_0$ be the set of all nonnegative integers. For points $P \in C$, we define the Weierstrass semigroup $H(P) \subset \mathbb{N}_0$ by

$$H(P) = \{\alpha \mid \text{there exists } f \in \mathcal{M}(C) \text{ with } (f)_\infty = \alpha P\}$$

where $(f)_\infty$ means the divisor of poles of f. Indeed, this set forms sub-semigroups of $\mathbb{N}_0$. The cardinality of the set $G(P) = \mathbb{N}_0 \setminus H(P)$ is exactly g and $G(P)$ is called by the gap sequence at P.

Equivalently, we can define Weierstrass point as follows; for any point $P \in C$, there exists a basis $\{\omega_i\}$, $(i = 1, \cdots, g)$ for the space of holomorphic differentials on C such that the order $\nu_P(\omega_i)$ of zero of ω_i on P satisfies

$$0 = \nu_P(\omega_1) < \nu_P(\omega_2) < \cdots < \nu_P(\omega_g) \leq 2g - 2.$$

The gap sequence at P is

$$1 = \nu_P(\omega_1) + 1 < \nu_P(\omega_2) + 1 < \cdots < \nu_P(\omega_g) + 1 \leq 2g - 1.$$

The Weierstrass weight at P is defined by

$$w(P) = \sum_{i=1}^{g} \Big(\nu_P(\omega_i) + 1 - i\Big)$$

and P is a Weierstrass point if $w(P) > 0$. There are only finitely many Weierstrass points on C. Thus by expanding differentials $\{\omega_i\}$ at a point P, we can calculate the Weierstrass weight and see whether P is a Weierstrass point or not (local method).

Another way to define Weierstrass point is to use the Wronskian for a basis of holomorphic differentials. The points at which the Wronskian vanishes are Weierstrass points and the order of zero at one of them is equal to the Weierstrass weight at the point. Thus from the Wronskian we can obtain information about the whole of Weierstrass point on C (Global method).

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For a more detailed introduction to Weierstrass points, see [GH], [Gunning].

We will introduce several problems containing Weierstrass points and list recent progress.

2. Numerical semi-group

The basic question regarding Weierstrass point, posed by Hurwitz[1893] is simply existence : Is every sub-semigroup H satisfying $\#(\mathbb{N}_0 \setminus H_P) = g$ an Weierstrass semigroup of some curve of genus g? But R.-O.Buchweitz showed that there exists a numerical semigroup which is not a Weierstrass semigroup([Buchweitz]).

Now the question is that which numerical semigroup is a Weierstrass semigroup? If a semigroup starts with 2, then the semigroup is the Weierstrass semigroup on a hyperelliptic curve. If a semigroup starts with 3, the following results are known : any curve admits at most two kinds of semigroup starting with 3, and for any semigroup starting with 3 there is a curve admitting that semigroup [Mac]. Komeda proved that for any semigroup starting with 4, there is a curve admitting that semigroup ([Komeda1]). On the other hand, Eisenbud and Harris proved that every semigroup of weight at most $g/2$ is a Weierstrass semigroup ([EH]). Kim and Komeda found some semigroups which are not Weierstrass semigroup and Torres gave some example of a symmetric numerical semigroups which cannot be realized as Weierstrass semigroup ([Kim4], [Torres]).

3. Weierstrass pairs

More recently, there was a beginning of the study of Weierstrass pairs of points of a Riemann surface C and of Weierstrass n sets of C. For points P, $Q \in C$, we define the Weierstrass semigroup of a pair of points $H(P,Q) \subset \mathbb{N}_0 \times \mathbb{N}_0$ by

$$H(P,Q) \quad = \quad \{(\alpha,\beta) \mid \text{there exists } f \in \mathcal{M}(C) \text{ with } (f)_\infty = \alpha P + \beta Q\}.$$

The set $G(P,Q) = \mathbb{N}_0 \times \mathbb{N}_0 \setminus H(P,Q)$ is finite, but its cardinality is dependent on the points P and Q. In [Kim2], the upper and lower bound of such sets are given as

$$\binom{g+2}{2} - 1 \leq card\ G(P,Q) \leq \binom{g+2}{2} - 1 - g - g^2.$$

In [BK], the authors defined the weight of a pair of points (P,Q) as

$$\begin{aligned} w(P,Q) &= \sum_{\alpha,\beta \geq 0} \Big(h^1\big(C, O_C(\alpha P + \beta Q)\big) - \max\big\{0,\ g - (\alpha+\beta)\big\}\Big) \\ &= \sum_{\alpha,\beta \geq 0} \Big(h^1\big(C, O_C(\alpha P + \beta Q)\big)\Big) - \binom{g+2}{3}, \end{aligned}$$

and they obtained the bound

$$0 \leq w(P,Q) \leq \frac{g^3 - g^2}{2},$$

where the last equality holds if and only if $|2P| = |2Q| = g^1_2$.

More generally, Weierstrass n-sets are defined in [BK] by points $P_i \in C$, $1 \leq i \leq n$, with $P_i \neq P_j$ for $i \neq j$ and such tha $\{P_1, \cdots, P_n\}$ has an exceptional behaviour with respect to the canonical series $|K_c|$ in the sense that

$$w(P_1, \cdots, P_n) = \sum w\Big(C, O_C(\sum_{1 \leq i \leq n} \alpha_i P_i)\Big) - \binom{g+n}{n+1} > 0.$$

Recently the structure of $H(P,Q)$ is more known and we can induce an information about Weierstrass point from the information on Weierstrass pairs.

For a γ-hyperelliptic curves and for points P, Q which lie in the same fibre, the partial structure of $H(P,Q)$ is given. Conversely, partial information on $H(P,Q)$ shows C is γ-hyperelliptic ([Kim5]).

4. Weierstrass semigroup and gonality

The complete list of the Weierstrass gap sequences of trigonal curves has been obtained in [Kim1] by proving that a numerical semigroup is the semigroup of some unramified Weierstrass point of some trigonal curves of genus $g \geq 5$ iff its gap sequence splits into two sequences of consecutive integers. The gap sequence of the points ramified with respect to the trigonal morphism have been computed earlier by Coppens([Coppens1], [Coppens2]). The gap sequence of a totally ramified point of a 4-gonal or 5-gonal curves is also found([Komeda1], [Komeda2]) but for a general points, it is open yet.

Also some possible gap sequence at a ramification points on cyclic coverings of an elliptic curves or on γ-hyperelliptic curves were studied in [Komeda4], [KuKo], [Torres].

5. Algebraic Geometric Code

Goppa realized that one could use the Riemann-Roch Theorem to show that certain codes produced from two divisor G and D on a curve have good properties([Goppa1], [Goppa2]). In particular, he gave lower bounds for the minimum distances of these codes. Gartia and Lax showed in [GL] that if G is taken to be a multiple of a point P, then knowledge of the gap at P may allow one to say that the minimum distance of the resulting code is greater than Goppa's lower bound. They and Kim developed this idea in [GKL] and showed that the presence of t consecutive gaps at P increase the minimum distance at least t than Goppa's lower bound.

Recently Lax generalized this idea again using Weierstrass pairs instesd of Weierstrass points([Lax]).

6. Moduli Space of Pointed Trigonal Curves

Also some work are given to find out the dimension of the moduli space of pointed curves with a prescribed gap sequences. Canuto give the dimension of the moduli space of the pointed trigonal curves of genus 5 ([Canuto]). The result was generalized by Stohr and Viana to an arbitrary genus. They studied the moduli space of the pointed trigonal curves with a prescribed gap sequence which splits into two consecutive parts. The existence of such gap sequence is guaranteed by Kim in [Kim1].

7. Multiple Weierstrass points

Weierstrass point is closely related with an automorphisms of the Riemann surface. Lewittes proved that if an automorphism of that Riemann surface has fixed points ≥ 5, then every fixed point is a Weierstrass point. A point P which lies in infinitely many of the set $\{P \in C : l(nD - sP) \geq 1\}$ is called a multiple D-Weierstrass point. Multiple Weierstrass points do exist. A number of authors have given criteria under which fixed points of non-trivial automorphisms are multiple

Weierstrass points in [Accola], [FK], [Guerrero], [Lewittes], [Takigawa], [HT]. Silverman and Voloch show that a curve of genus ≥ 2 yield only finitely many multiple Weirstrass points and in fact rather rare ([SV]).

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