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EMBEDDED SURFACES IN 4-MANIFOLDS AND SEIBERG-WITTEN INVARIANT

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Abstract. We survey the problem of representing a 2-dimensional homology class σ by a smoothly embedded, oriented surface Σ of minimal complexity. Applying Seiberg-Witten invariant, we can achieve a lower bound for the genus of embedded surface, which is called adjunction inequality. We discuss the existence of minimal genus embedding sharpening the inequality and its relation to the Seiberg-Witten invariant.

1. Introduction

Given a 2-dimensional homology class σ in a smooth 4-manifold X, what is the least possible genus for a smoothly embedded, oriented surface Σ in X whose fundamental class is σ? Gauge theory has been a successful tool in answering a collection of basic question of this sort. The introduction of the Seiberg-Witten monopole equations and the replacement of Donaldson's polynomial invariants by apparently equivalent monopole invariants [W] leads to much simpler proofs of essentially the same results. To give just one example, if X is a smooth quintic surface in $\mathbf{C}P^3$ and Σ is a smooth algebraic curve obtained as the intersection of X with any other complex surface $H \subset \mathbf{C}P^3$, then Σ is known to achieve the smallest possible genus in its homology class. This result is known to hold for a complex curve in complex algebraic surfaces in general, and the theorems can be extended to the case of symplectic manifolds[T1]. Although gauge theory gives lower bounds on the genus of embedded surfaces in general 4-manifolds, these lower bounds should no longer be expected to be sharp, at least in the form in which they are usually phrased. We will focus on finding homology classes which sharpen the lower bound of embedded genus so called adjunction inequality in a symplectic 4-manifold and discuss the relation between minimal genus embedding and the Seiberg-Witten invariants.

2. Seiberg-Witten equation and Adjunction inequality

To discuss the lower bound for the genus of embedded surface, we review the Seiberg-Witten monopole equations and the adjunction inequality. Most of material can be found in [M, T1,W].

The Seiberg-Witten equations: Let X be an oriented, Riemannian 4-manifold.

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and Let $\mathcal{L}$ be a characteristic line bundle in other words, $c_1(\mathcal{L}) \equiv w_2(X) \bmod(2)$. This determines a *spin*^{c} structure on X. Fix a *spin*^{c} structure, there are associated complex vector bundles called complex spinor bundle, $S(\mathcal{L}) = S^+(\mathcal{L}) \oplus S^-(\mathcal{L})$, where $S^{\pm}(\mathcal{L})$ is $U(2)$ positive(negative) complex spinor bundle.

We consider that the Seiberg-Witten equations constitute a system of partial differential equations for a pair (A, ϕ), where A is a Hermitian connection on the complex line bundle $\mathcal{L}$ and where ϕ is a section of S^+. These equations read;

- $\not{D}_A \phi = 0$
- $F_A^+ = q(\phi)$,

where $\not{D}_A$ is the Dirac operator as defined using the connection A on $\mathcal{L}$ and the Levi-Civita connection on TX. F_A^+ is the self-dual part of curvature F_A and $q(\phi) = \phi \otimes \phi^* - \frac{1}{2}(|\phi|^2)Id$ is the quadratic map from S^+ to $i\Lambda^+$ which can be understood as trace free hermitian endomorphism of S^+, and μ is a purely imaginary self-dual two form.

The Seiberg-Witten invariants: The moduli space $\mathcal{M}_X(\mathcal{L})$ is the space of solution (A, ϕ) modulo the action of gauge group $C^\infty(X; S^1)$. We can also perturb the equations, by adding an arbitrary self-dual 2-form μ in the second part of the equation. We write $\mathcal{M}_X(\mathcal{L}, \mu)$ for the solution space. The equations have an index which is called as a formal dimension of SW moduli space for *spin*^{c} structure $\mathcal{L}$.

$$d(\mathcal{L}) = \frac{1}{4}(c_1(\mathcal{L}) \cdot c_1(\mathcal{L}) - 2e(X) - 3\sigma(X)).$$

The basic facts about the moduli space are these: *The moduli space $\mathcal{M}(\mathcal{L}, \mu)$ is compact. For an open, dense set of perturbations μ, the moduli space is consists of irreducible solutions ($\phi \neq 0$) and it is a smooth manifold of dimension $d(\mathcal{L})$.*

Let $\mathcal{L}$ be a *spin*^{c} structure with $d(\mathcal{L}) = 0$, so that $c_1(\mathcal{L}) \cdot c_1(\mathcal{L}) = 2e(X) + 3\sigma(X)$. Suppose that X has $b_2^+(X) \geq 2$ then choose a generic μ to make $\mathcal{M}_X(\mathcal{L}, \mu)$ a smooth manifold. Then this $\mathcal{M}_X(\mathcal{L})$ is a finite union of signed points and the Seiberg-Witten invariant is the sum over these points of the corresponding ± 1's (The sign depends on the orientation). The number of solutions, counted with suitable signs, is independent of choice of perturbation and the choice of metric. We write

$$SW_X(\mathcal{L}) \in \mathbf{Z}$$

for this number. This is the Seiberg-Witten monopole invariant for X with *spin*^{c} structure $\mathcal{L}$. We also define the Seiberg-Witten invariants for the *spin*^{c} structure with positive index [M, W]. We call the characteristic class $\mathcal{L}$ to be a *Seiberg-Witten basic class* (for brevity, *basic class*) for X if $SW_X(\mathcal{L}) \neq 0$.

The first significant result about the basic class was proved by Witten in [W]. This was the statement that, for a smooth algebraic surface with $b^+ > 2$, (e.g. a hypersurface in $\mathbf{C}P^3$ of degree 4 of more), the first Chern class and its negative (canonical class) are basic classes. This was soon generalized by Taubes in [T1,T2] to symplectic manifolds, in the following form. A symplectic structure ω on a manifold determines an almost-complex structure uniquely up to deformation. The *canonical class* K_ω is $-c_1(\omega)$.

Theorem 2.1. *(Taubes [T1], [T2]). Let (X, ω) be a compact symplectic 4-manifold. Suppose either $b^+ > 1$, or that $b^+ = 1$ and $K_\omega \cup \omega$ and K_ω^2 are both positive. The canonical class is a basic class.*

Up to now, it is found that there are simply connected 4-manifolds having *SW-basic class* which cannot carry any symplectic structure. Nevertheless, it is observed that every basic class for the known examples with $b^+ > 1$ has zero formal dimension i.e. $SW_X(\mathcal{L}) \neq 0$ implies that $d(\mathcal{L}) = \frac{1}{4}(c_1(\mathcal{L})^2 - 3\sigma(X) - 2e(X)) = 0$. This imposes some restrictions on the basic classes , which says that every basic class defines a almost complex structure on X. It is called that X is of *Seiberg-Witten simple type* if $d(\mathcal{L}) > 0$ then $SW_X(\mathcal{L}) = 0$. It is conjectured that X is SW-simple type for the general 4-manifold X with $b_2^+(X) > 1$. We call it simple-type conjecture briefly. Furthermore, suppose X has a SW-basic class, this gives a minimal genus bound for the embedded surface so called *Adjunction inequality.* It also gives an upper bound for a self-intersection number of the embedded surface of genus g and intersection number of surface with a SW-basic class. We states the adjunction inequality:

Theorem 2.2. *(Adjunction inequality* [D1, KM2, MST] *Suppose X is a smooth, oriented, closed 4-manifold with $b_2^+(X) > 1$, let $\mathcal{L}$ be a basic class and Σ is an embedded connected oriented surface representing nontrivial homology class $\sigma \neq 0$. If $\sigma \cdot \sigma \geq 0$, then*

$$2g(\Sigma) - 2 \geq \sigma \cdot \sigma + | < \mathcal{L}, \Sigma > |,$$

where $g(\Sigma)$ is the genus of Σ.

The proof of this theorem only requires that solutions for the Seiberg-Witten equation for some *spin*^{c} structure $\mathcal{L}$ is non-empty for every choice of Riemannian metric.

Remark 2.3. *If X is of SW simple type, the above adjunction formula also holds for the homology class σ of negative intersection* [OS]. *However we know very little about classes of negative square even in algebraic surface, for example.*

It is easy to see that existence of an embedded sphere (Riemann surface of genus 0) of nonnegative intersection imply that X has vanishing Seiberg-Witten invariants [MST]. Suppose X contains an embedded surface of genus g whose self-intersection number is strictly greater than $2g-2$ then we can conclude that X has no solutions for the Seiberg-Witten equation. Furthermore, if $\Sigma_g \cdot \Sigma_g = 2g-2$ for $g \geq 0$ the we can show that X is of SW-simple type. In any way, we can prove the following corollary.

Corollary 2.4. *Let X be a smooth 4-manifold with $b_2^+(X) \geq 2$ and $\Sigma_g \hookrightarrow X$ be an embedded Riemann surface of genus $g \geq 1$ in X and $[\Sigma_g] \neq 0 \in H_2(X, \mathbf{Z})$*

1) *If $\Sigma_g \cdot \Sigma_g > 2g-2$, then X has vanishing Seiberg-Witten invariants.*
2) *If $\Sigma_g \cdot \Sigma_g = 2g-2$, then X is of Seiberg-Witten simple-type.*

Proof: The first case follows by the adjunction formula. If Σ_g has intersection number $2g-2$ and X has a basic class, then it is clearly a minimal genus representative in its homology class by the adjunction formula. For the case $g > 1$, if a basic class $\mathcal{L}$ has positive dimensional moduli space, then $\mathcal{L} + 3E$ becomes a basic class for the blow-up manifold $X \# \overline{\mathbf{C}P^2}$, where E is the Poincare-dual of the exceptional sphere. It follows that the formal dimension for the *spin*^{c} structure $\mathcal{L} + 3E$ is nonnegative, i.e. $d(\mathcal{L} + 3E) = d(\mathcal{L} + E) - 2 = d(\mathcal{L}) - 2 \geq 0$. Let $\tilde{\Sigma}_g$ be an internal connected sum with Σ_g and exceptional sphere E. It implies the homology class $\tilde{\Sigma}_g = \Sigma_g - E$. By applying the adjunction formula for $\mathcal{L} + 3E$ and $\tilde{\Sigma}_g$, then

$$| < (\mathcal{L} + 3E), \tilde{\Sigma}_g > | + \tilde{\Sigma}_g \cdot \tilde{\Sigma}_g = 3 + (2g-3) = 2g > 2g-2.$$

It draws a contradiction. Hence $d(\mathcal{L})$ must be zero.

Suppose X has an embedded torus of trivial square, by applying the product formula along the boundary of small tubular neighborhood of the torus, it is proved that X is of Seiberg-Witten simple type [MMS2].

Remark 2.5. *By the same way of proving the corollary, we can also prove that if SW-basic class $\mathcal{L}$ sharpens the adjunction inequality for some embedded surface of positive intersection number then the dimension of the SW-moduli space $d(\mathcal{L}) = 0$.*

3. Minimal genus embedding

Since there are manifolds such as $\mathbf{C}P^2\#\mathbf{C}P^2$, where the gauge theory invariants have told us very little, we will consider the 4-manifolds having a SW basic class.
Symplectic submanifolds or complex curves: By the Taubes's result, every symplectic 4-manifolds has basic class arise as K_ω for some symplectic form. The adjunction inequality is an equality for symplectic submanifolds, or smooth algebraic curves in a complex surface, where it is usually referred to as the adjunction formula. For example, the 4-torus, where the $spin^c$ structure K_{T^4} with $c_1(K_{T^4}) = 0$ has monopole invariant $SW_{T^4}(K_{T^4}) = 1$, we learn that embedded surface in T^4 satisfy $2g(\Sigma) - 2 \geq |\Sigma \cdot \Sigma|$. (The absolute value on the right-hand side because T^4 looks same with either orientation). The result for T^4 above is sharp: every class σ can be represented by a surface of complexity $|\sigma \cdot \sigma|$. More generally, if X carries a symplectic form ω, then the adjunction inequality is sharp for at least a significant range of classes σ, by Donaldson's theorem [D2] on the existence of symplectic submanifolds.That is, since non-degeneracy of a closed 2-form is an open condition, any rational cohomology class $\Omega' \in H^2(X;\mathbf{Q})$ sufficiently close to the class $\Omega = [\omega]$ in $H^2(X;\mathbf{R})$ is represented by a symplectic form ω', and the theorem[D2] says that for some large k (depending on Ω') an integer class $\mathrm{PD}(k\Omega')$ is represented by a symplectic submanifold Σ, for which the adjunction inequality is inevitably an equality.

Remark 3.1. *The lagrangian submanifolds in a symplectic submanifolds also shapen the inequality. Here, $\Sigma \subset (X, \omega)$ is a lagrangian submanifold if $\omega|_\Sigma \equiv 0$. There is a way to make lagrangian surface to be symplectic submanifold by changing the symplectic form to new one.*

To find a lagrangian embedding in a symplectic 4-manifolds is quite subtle problem. For example, we can construct a large number of lagrangian surfaces of high genus in a smooth hypersurface of degree greater than 4 [CJ]. We raise a question concerning about the existence of lagrangian embedding in a symplectic 4-manifold.

Question 3.2. *Let (X, ω) be a symplectic 4-manifold with $b^+ \geq 2$. Does X always contains a lagrangian embedding of higher genus (≥ 2) ?*

Suppose 2-homology class σ can be represented by a lagrangian submanifold Σ then It necessarily have zero pairing with the canonical class and symplectic form, i.e. $< K_\omega, \sigma >=< \omega, \sigma >= 0$. We can find such a lagrangian embedding case by case, as a fixed locus of anti-symplectic involution, etc.
Minimal genus embedding for multiple of given class : Suppose a homology class σ with $\sigma^2 \geq 0$ shapen the adjunction inequality then we can find that $k\sigma$ has an embedded surface Σ_k to make the adjunction inequality equal. We have $g(\Sigma_k) = \frac{1}{2}(k^2 - k)\sigma^2 + g(\Sigma)$, where Σ is an minimal genus representative of σ. This

follows from a simple observation that suppose Σ_1 and Σ_2 be embedded surfaces which meet transverse and positively at $\sigma_1 \cdot \sigma_2$ distinct points then we do surgery at each intersection point by replacing the one point union of 2-disk $D_1^2 \cup D_2^2$ with a 2-cylinder $S^2 \times D^1$. The resulting surface has genus $g(\Sigma_1) + g(\Sigma_2) + \Sigma_1 \cdot \Sigma_2 - 1$. Let Σ be the minimal representative of σ then we can generate $k-1$ embedded surfaces near the tubular neighborhood, where for each pair intersects tranverse at distinct σ^2 points. Perform a surgery at each intersection point then we get a minimal genus representative Σ_k of $k\sigma$. The above assertion follows from it. Whether the converse of this process is possible or not is now in question.

Question 3.3. *Let σ be a 2-homology class with $\sigma^2 > 0$ in a symplectic 4-manifold (X, ω). Suppose $k\sigma$ sharpens the adjunction inequality for $k \geq 2$, Does σ also shapen it ?*

For example, it gives rise to a question whether any ample class in algebraic surface can be represented as a minimal genus embedding which satisfies the adjunction formula. There are many other questions related this minimal genus problems. It is suggested that reader may refer the survey paper by P. Kronheimer [K] which is available at http://math.harvard.edu/ kronheim.

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