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SOME RECENT PROGRESS IN FUJITA CONJECTURE

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Abstract. Linear systems have been one of the primary tools to understand the geometry of algebraic varieties. However, for the last decade there has been two new developments. First, vector bundle technique has emerged as a powerful tool to analyze the linear systems on surfaces. Second, the cohomological method developed to study higher dimensional varieties has found its applications on linear systems on threefold and higher. This article is an survey on this developments with some recent results.

Fujita Conjecture

Let X be a smooth projective variety and let H be an ample Cartier divisor on X. It is an interesting and classical question when the adjoint linear system $K_X + H$ is base point free or even very ample where K_X is the canonical divisor on X. Thanks to Serre's theorem, we know that if H is *sufficiently positive*, e.g. $H = nA$ where A is ample and $n >> 0$, then the above linear systems is very ample. But the result is not so practical. For this, we consider two examples.

(1) First, it is an simple application of the classical Riemann-Roch theorem that an Cartier divisor D on a smooth projective curve C is base point free if $deg(D) \geq 2g$ and very ample if $deg(D) \geq 2g+1$ where g is the genus of the curve C.

(2) As a second example, we consider the projective n-space $\mathbb{P}^n$ and a hyperplane section H. Since $K_{\mathbb{P}^n} = -(n+1)H$, $K_{\mathbb{P}^n} + mH$ is base point free if $m \geq n+1$ and very ample if $m \geq n+2$.

In view of (2), we could restate (1) as follows

(1') Let C be a smooth projective curve and let D be an effective divisor on C. Then $K_C + D$ is base point free if $deg(D) \geq 2$ and very ample if $def(D) \geq 3$.

Generalizing these two example, Fujita raised in [F1] the following conjecture.

Conjecture A. *Let X be a smooth projective variety of dimension n, let K_X be a canonical divisor on X, and let H be an ample Cartier divisor on X. Then*

(BP) *$|K_X + tH|$ is base point free for $t \geq n+1$.*

(VA) *$|K_X + tH|$ is very ample for $t \geq n+2$.* □

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Surfaces and Reider's theorem

As for surface, we know that the above conjecture is true, due to the following beautiful theorem of Reider.

Theorem B [Rei]. *Let S be a smooth projective surface and H be a nef Cartier divisor on S.*

(BP) *Assume that $H^2 \geq 5$ and $|K_S + H|$ has a base point at $p \in S$. Then there exists an effective divisor C on S containing p such that*

$$H \cdot C = 1 \quad \textit{and} \quad C^2 = 0; \qquad \textit{or}$$
$$H \cdot C = 0 \quad \textit{and} \quad C^2 = -1.$$

(VA) *If $H \geq 10$ and if p, q are two (possibly infinitely near) points in S which fails to be separated by $|K_S + H|$, then there exists an curve C on S containing p and q such that*

$$H \cdot C = 2 \quad \textit{and} \quad C^2 = 0; \qquad \textit{or}$$
$$H \cdot C = 1 \quad \textit{and} \quad C^2 = 0 \quad \textit{or} \quad -1; \qquad \textit{or}$$
$$H \cdot C = 0 \quad \textit{and} \quad C^2 = -1 \quad \textit{or} \quad -2. \Box$$

Main ingredients of the proof is Bogomolov's instability theorem and Hodge index theorem. If one wants to prove weaker result as in Conjecture A, Hodge index theorem is not necessary. In fact, Fernández del Busto [FdB] proved that Bogomolov's instability theorem follows from Kawamata-Viehweg vanishing theorem, and a direct proof of Conjecture A using Kawamata-Viehweg vanishing theorem and graded linear systems technique(cf.[ELN]) is given in [Maş],[L2]. We recommend the lecture note [Laz] of Lazarsfeld for a very nice introduction on this circle of area.

As for the pluricanonical systems on surface it has been extensively studied by Kodaira (cf. [Kod]) and Bombieri (cf. [Bom]) in 60's and 70's. And now together with Theorem B, the picture is clear for high power of canonical systems. However below the Reider's range less is known. Next we collect some of the results(cf. [Bom],[Ca1],
[CC1],[CC2],[Fr],[Re]).

Theorem C. *Let S be a smooth projective minimal surface of general type and let S_0 be its canonical model. Then*

(BP) (1) *$|mK_{S_0}|$ is base point free for $m \geq 3$.*

(2) *$|2K_{S_0}|$ is base point free if $K_{S_0}^2 \geq 5$ or $p_g(S_0) > 0$.*

(VA) (1) *$|mK_{S_0}|$ is very ample for $m \geq 5$.*

(2) *$|4K_{S_0}|$(resp.$|3K_{S_0}|$) is very ample if $K_{S_0}^2 \geq 2$(resp.$K_{S_0}^2 \geq 3$).* $\Box$

Theorem C essentially follows from Theorem B and Adjunction theorem except the condition $p_g(S_0) > 0$ in (2) of (BP), which is a consequence of results independently obtained by various authors. We refer to [Ca2] and [Ca3] for the detailed account on bicanonical system and canonical systems on projective minimal surface of general type.

Higher dimensional varieties

Here we explain developments in higher dimensional varieties. We recommend two lecture notes by Ein [E] and Kollár [Ko2] for overviews on this area.

Since the beautiful result of Reider, there has been numerous efforts to extend it to higher dimensional varieties. The first break through came with the work [D1] of Demailly. He used analytic method to obtain effective results on double adjoint linear systems on algebraic varieties. Then Kollár [Ko1] gave a similar result using algebraic method. In both case the bound was too large to be practical. Later Angehrn and Siu [AS] was able to lower the bound substantially by providing a quadratic bound for base point freeness of adjoint linear systems on smooth algebraic varieties using a *limit argument*. Their method was later translated in algebraic setting by Kollár(cf. [Ko2]).

As for the practical results, Ein and Lazarsfeld [EL1] was able to prove (BP) of Conjecture A for threefolds using *Kawamata-Viehweg vanishing theorem*. Their idea was to create a Q-divisor D with a high multiplicity at a point using Riemann Roch theorem and then apply Kawamata-Viehweg vanishing theorem for Q-divisor. If D is less singular out side of that point then it immediately implies the freeness. Otherwise, they reduce the problem to lower dimensional variety(they called it the critical variety) which appears as the base locus of certain graded linear systems. Following their remarkable results of Ein and Lazarsfeld, there were two following development.

First, Kawamata [Ka2] showed that the critical varieties are rational. Using this, he was able to prove (BP) of Conjecture A for 4-folds in [Ka1].

Second, Fujita [F2] observed that if we indeed have D which have high multiplicity alone positive dimensional variety, then we can choose D with the multiplicity much higher than Riemann-Roch theorem predicts. This line of idea was systematically developed by Ein and Lazarsfeld [EL2], Helmke [H1] [H2], and Kawamata [Ka1]. Specially, Helmke [H1] was able to substantially improve the result of Angehrn and Siu, and proved the following

Theorem D. *Let X be a smooth projective varieties, let p be a point in X, and let H be an nef divisor on X such that*

(1) $H^n > n$.

(2) $H \cdot D \geq n$ *for all divisor* D *through* p.

(3) $H \cdot W \geq \binom{n-1}{d-1} n$ *for all subvariety* W *containing* p *with* $mult_p W \leq \binom{n-1}{d-1}$.

Then $|K_X + H|$ *is base point free at* p. □

Meanwhile, Ein and Lazarsfeld observed that for the induction process, it is more natural to consider a larger category of varieties rather than categories of smooth varieties, and to deal with Q-divisors rather Cartier divisors. And they proposed the following conjecture which implies (BP) of Conjecture A(cf.[EL2],[L3]).

Conjecture E. *Let X be a normal projective variety of dimension n. Let Δ be an effective Q-divisor on X such that (X, Δ) is Kawamata log terminal. Let A be an Q-ample divisor on X such that $K_X + \Delta + A$ is Cartier. Let $\sigma_1, \dots, \sigma_n$ be rational numbers satisfying the following three conditions.*

(1) $A^n \geq \sigma_n^n$ *and* $A^p \cdot W \geq \sigma_p^p$ *for all subvariety* W *in* X *with* $dim W = p$.

(2) $\sigma_n > n$ *and* $\sigma_n \geq \sigma_{n-1} \geq \cdots \geq \sigma_2$.

(3) $\sigma_p(1 - \frac{1}{\sigma_n} - \cdots \frac{1}{\sigma_{p+1}}) \geq p$, *where* $p = 1, \ldots, n-1$.

Then $|K_X + \Delta + A|$ *is base point free.* □

So far it has been verified up to $n = 3$(cf. [L3]).

As for the pluricanonical systems on threefolds, we have the following theorem, which is a consequence of works of several authors (cf. [EL1],[ELM],[M],[L1],[L3],[Ch]).

Theorem F. *Let* X *be projective canonical threefolds with ample Cartier canonical divisor* K_X. *Then*

(1) $|mK_X|$ *is base point free for* $m \geq 4$.

(2) $|mK_X|$ *separates two distinct point for* $m \geq 6$.

(3) $|5K_X|$ *induce birational morphism.*

(4) *Assume that* X *is smooth. Then* $|mK_X|$ *is very ample for* $m \geq 10$. □

Sketch of Proof

Here we give a very rough idea in the proofs of the results presented above by sketching the proof of Theorem D.

First, we need to recall some properties of multiplier ideals. Now there are several papers available on this subject including [E], [Kaw2], and [Kol1]. Here we take [E] as the main reference. There things are presented in smooth setting, but with minor modifications, they can be extended to log-terminal setting. Now we begin the discussion on the multiplier ideals.

Multiplier ideals and Critical varieties. Let X be a log terminal variety, let G be an effective Q-Cartier divisor on X, and let $f : Y \to X$ be an embedded resolution of G. Define b_i and g_i by

$$K_Y - f^* K_X = \sum b_i F_i$$

and

$$f^* G = \sum g_i F_i,$$

where F_i's are the distinct irreducible smooth divisors in simple normal crossing. Let

$$\sum \lceil (b_i - g_i) \rceil F_i = P - N,$$

where P and N are effective divisors with no common components. Since P is f-exceptional, by Lemma 1-3-2 in [KMM],

$$f_* \mathcal{O}_P(P - N) = f_* \mathcal{O}_P(P) = 0.$$

So we obtain an ideal sheaf

$$f_* \mathcal{O}_Y(P - N) = f_* \mathcal{O}_Y(-N) \subset \mathcal{O}_X.$$

We will keep these notations and assumptions.

Proposition G. *$f_*\mathcal{O}_Y(P-N)$ is called the multiplier ideal of G. Let $Z(G)$ be the scheme defined by this ideal and we will note the multiplier ideal by $\mathcal{I}_{Z(G)}$. By some fairly standard method, one can check that $\mathcal{I}_{Z(G)}$ is independent of the choice of the embedded resolutions.* □

Definition H. *Let p be a point in X. We say that G is critical at p if*
(1) $p \in Z(G)$ but $p \notin Z((1-\epsilon)G)$ for any $\epsilon > 0$.
(2) For all sufficiently small $0 < \epsilon << 1$,

$$\lceil K_Y - f^*(K_X + G)\rceil = \lceil K_Y - f^*(K_X + (1-\epsilon)G)\rceil - F_i$$

for an unique F_i. Note that such F_i should intersect $f^{-1}(p)$ by (1). We call F_i the critical component and $f(F_i)$ the critical variety for G at p. □

A General Method and Invariance of Liftings. Here we state two key ingredients for the proof of the base point freeness of adjoint linear series in general.

Proposition I. *Let p be a point in X, and we assume that G is critical at p with a 0-dimensional critical variety. If H is a Cartier divisor on X such that $H - K_X - G$ is nef and big, then $|H|$ is free at p.* □

Proposition J. *Suppose G is critical at p with the critical variety Z. Let B be an effective Q-Cartier divisor on Z. Let D_1 and D_2 be two liftings of B. If*

$$W \subset Z((1-s)D_1 + (1-t)G)$$

for all sufficiently small s and t, then

$$W \subset Z((1-s)D_2 + (1-t)G)$$

for all sufficiently small s and t. □

Definition K. *Suppose G is critical at p with the critical variety Z. Let B be an effective Q-Cartier divisor on Z. An effective Q-Cartier divisor D is said to be a nice lifting of B with respect to G if $D|_Z = B$ and*

$$Z((1-t)G + D|_{X-Z}) = Z((1-t)G|_{X-Z})$$

for $0 < t << 1$. □

Proposition L. *Suppose G is critical at p with the critical variety Z. Let B be an effective Q-Cartier divisor on Z. Let A be a Q-Cartier ample divisor on X such that B is Q-linearly equivalent to $A|_Z$. Then there is a nice lifting D of B with respect to G such that D is Q-linearly equivalent to A.* □

Sketch of the proof of Theorem D. Here we keep the same notation as in theorem D.

We start by creating a Q-Cartier divisor D, Q-linearly equivalent to H, so that $ord_p D > n$ using Riemann-Roch theorem. Then by examining the coefficient of the exceptional divisor of the blowing up at p, we see that there is a positive constant $\lambda < 1$ such that λD is critical at p. (Here we might need to perturb λD a little bit to make the second condition of Definition H holds. but it is only technical problem. So for the simplicity, here we assume that it already holds.) Let Z be the critical variety of λD. If Z is a point, then it follows from Proposition I that $K_X + H$ is free at p. So we assume that $d := dim Z > 0$.

In this case, Helmke [H2] has shown that $mult_p Z \le \binom{n-1}{d-1}$. Then from Riemann-Roch theorem, we create a Q-divisor B, Q-linearly equivalent to $H|_Z$ such that $ord_p B \ge n$. We have the following sequences for all $k > 0$.

$$\mathcal{I}^k_{p/X} \longrightarrow \mathcal{I}^k_{p/Z} \longrightarrow 0$$

where $\mathcal{I}_{p/X}$(resp.$\mathcal{I}_{p/Z}$) is the sheaf of the maximal ideals at p in $\mathcal{O}_X$(resp, $\mathcal{O}_Z$). Thus locally there is a lifting D' of B such that $ord_p D' \ge n$. Let λ' be the smallest number such that $p \in Z(\lambda D + \lambda' D')$. Again we assume that $\lambda D + \lambda' D'$ is critical at p. With some analysis on the discrepancy (cf. [E]) of the exceptional divisors, one can show that $\lambda + \lambda' < 1$. Let D'' be a nice lifting of B with respect to λD. Then from Theorem J, $p \in Z(\lambda D + \lambda' D'')$. Now if the critical variety of $\lambda D + \lambda' D''$ is a point, then since $\lambda + \lambda' < 1$ $K_X + H$ is free at p. Otherwise we have to repeat above process. But it must stop in at most n-steps. □

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