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k-GONAL CURVES

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Abstract. The gonality of algebraic curve is very important numerical invariant to study algebraic curves. In this note, we introduce some results about the k-gonal curves including ours. Precisely, they are the results related with linear series on k-gonal curves, subvarieties of Jacobian variety, Clifford indices and projectively normal embeddings of k-gonal curves and the reducibility of the Hilbert scheme of curves of degree d and genus g in $\boldsymbol{P}^r$(; from the exitences of components given by the k-gonal curves).

1. Introduction

Let C be a smooth irreducible projective algebraic curve of genus g over the field of complex numbers. The gonality of C is defined by

$$\text{gon }(C) = \min\{n \mid \text{ there is a } \ g^1_n \ \text{ on } \ C\}.$$

The Brill-Noether theory says $\text{gon}(C) \leq [\frac{g+3}{2}]$. Let $\mathcal{M}^1_{g,k}$ be the closure of the locus of $k-$gonal curves in the moduli space $\mathcal{M}_g$ of smooth curves of genus g. Then it is irreducible of dimension $2g-5+2k$ and the curve corresponding to its general point has a unique pencil of degree k.([AC1]) Those facts have been basic about $k-$gonal curves. Anyway, there are many other results about them and so it is probabable that for a given curve we get the range of its gonality and characterize the curve. That is one of the general methods to study curves. Thus it is valuable to obtain informations about $k-$gonal curves.

In this note, we introduce some basic results about the k-gonal curves and recent ones related with linear series on k-gonal curves, subvarieties of Jacobian variety, the Clifford indices and projectively normal embeddings of k-gonal curves and the reducibility of the Hilbert scheme of curves of degree d and genus g in $\boldsymbol{P}^r$(; from the exitences of components given by the k-gonal curves).

2. Basic results

Let C be a $k-$gonal curve of genus $g > 0$. Then it has known that $2 \leq k \leq [(g+3)/2]$ by the Brill-Noether theory. In case $k = 2$, that is hyperelliptic case, any linear series g^r_d, $d \leq 2g-2$, is compounded of g^1_2(;$g^r_d = rg^1_2 + B$, B : base locus) and C is birationally isomorphic to plane curve given by the special equation.

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Thus the hyperelliptic curves has been well characerized. In the maximal case $k = [(g+3)/2]$(; C : general genus g curve), C has base point free special pencils of degree $n > k$ which are not compounded of g^1_k. For the other cases, we have the folllowing theorem of Abarello and Cornalba which has been used in the proofs of many theorems about $k-$gonal curves.

Theorem 2.1. *([AC1]) Let C be a general k-gonal curve of genus g, $2 \leq k \leq [(g+1)/2]$. Then C possesses only one g^1_k and any g^1_n, $n \leq [(g+1)/2]$, is compounded of the unique g^1_k.*

We recently got a generalized result, which we will state in the following section.

Let C be a smooth curve. Then there is an important object $W^r_d(C)$ which is defined by the locus in $J(C)$ corresponding to those line bundles of degree d with at most $r+1$ linearly independent global sections. Using this object, we see the kind and family of linear series which C can have, and sometimes characterize the curve C. The followings are representative theorems about this object.

Theorem 2.2. *([KL], [GH]) Let the Brill-Noether number $\rho(d,g,r) = g-(r+1)(g-d+r)$. Then we have $dimW^r_d(C) \geq \rho(d,g,r)$ for any curve C of genus g. If C is a general curve of genus g, then $dimW^r_d(C) = \rho(d,g,r)$ in case $\rho(d,g,r) \geq 0$ and $W^r_d(C) = \phi$ in case $\rho(d,g,r) < 0$.*

This theorem is a fundamental theory to study curves. In case of a general k-gonal curve, Coppens and G. Martens recently proved the similar one to the latter part of the above in some range of d and r, which will be stated in the following section.

Theorem 2.3. *([M1]) Let C be a smooth curve of genus $g \geq 3$. Let d, r be intergers such that $d \leq g+r-2$, $r \geq 1$. Then, $dimW^r_d(C) \leq d-2r$ and equality holds if and only if C is hyperelliptic.*

Theorem 2.3. has been generalized in many directions by Mumford, Keem and many others. We will introduce these kinds of recent results for $k-$gonal curves in the following section.

We also have another important numerical invariant Clifford index of a smooth algebraic curve C of genus g, which is given by

$$\text{cliff}(C) = \min\{\text{cliff}(D) \mid \text{deg}D \leq g-1\}.$$

where cliff(D) =deg$D$ $-$ 2dim$|D|$. In both mininmal and maximal cases(; $k=2$, $k = [\frac{g+3}{2}]$) of the gonalities k of genus g curves, we have cliff(C) =gon$(C) - 2$ for a general k-gonal curve C by Clifford's Thoerem and Theorem 2.2. But the equality also holds for a general k-gonal curve C for any case of gonality k.([B1], [KK]) Moreover, Cliffford index of C is computed by only the uinique g^1_k on C. The more stronger fact is given by the generalized result of Theorem 2.1. which is mentioned in the bellow of Theorem 2.1.

3. Linear Series on k-gonal curves

In this section, we introduce recent progresses of results about $k-$gonal curves of genus g. First we start with Ballico's theorem.

Theorem 3.1. *([B2]) Let C be a general $k-$gonal curve of genus g and let g^1_k be the unique pencil of degree k on C. Then $dim|rg^1_k| = r$ for $r \leq [\frac{g}{k-1}]$ and $dim|rg^1_k| > r$ for $r > [\frac{g}{k-1}]$.*

A base point free complete linear series $|D|$ on C is said to be primitive if its residual series $|K_C - D|$ is also base point free, where K_C denotes the canonical divisor of C. Any curve C has trival primitive linear series g^0_0 and $|K_C|$. And if $g \geq 4$, then C has nontrivial primitive linear series. We have an interest in primitive linear series, since any non-primitive linear series becomes primitive by removing base locus.

Theorem 3.2. *([CKM1]) Let g^1_k be a complete base point free pencil on C and let $dim|rg^1_k| = r$ for $r = 1, 2, \cdots, t$. Then the series g^1_k, $2g^1_k, \cdots, (t-1)g^1_k$ are primitive.*

Using two above theorems, a general $k-$gonal curve has many primitive linear series given by multiples of the unique g^1_k. They also proved in another paper that a general k-gonal curve, $3 < k < [\frac{g+3}{2}]$, has primitive pencil g^1_n if $\rho(n, g, 1) \geq 0$(; equavalently $n \geq [\frac{g+3}{2}]$) by the following in the same paper.

Theorem 3.3. *([CKM2]) Let C be a general $k-$gonal curve of genus g, $3 \leq k < [\frac{g+3}{2}]$. Then $W^1_n \neq W^1_k + W_{n-k}$ $(n \leq g)$ if $\rho(n, g, 1) \geq 0$.*

The above primitive pencils on the general k-gonal C of genus g have Clifford indices at least $[\frac{g-1}{2}]$. Thus they has conjectured that the general $k-$gonal curve C of genus g, $3 < k < [\frac{g+3}{2}]$, any primitive linear series $|D|$'s are multiples of the unique g^1_k if cliff$(D) < [\frac{g-1}{2}]$. This almost solved by the following, which is a sort of generalized result of Theorem 2.1.

Theorem 3.4. *([K1]) Let C be the general k-gonal curve of genus $g \geq 4$, $k \geq 4$, and $|F|$ the unique pencil of degree k on C. If C has a linear series $|D|$ of cliff$(D) \leq [(g-4)/2]$ and $degD \leq g-1$, then $|D|$ is compounded of $|F|$.*

Recently, Coppens and G. Martens obtained a kind of Brill-Noether theory for the moduli space of smooth $k-$gonal curves of genus g, which is the similar result for $k-$gonal curves to Theorem 2.2.

Theorem 3.5. *([CM]) Let C be a general $k-$gonal curve of genus g. Let r, d be positive integers such that $d - g < r \leq k - 2$ and $\rho(d, g, r) \geq 0$. Then the variety $W^r_d(C)$ of special linear series on C has an irreducible component of the expected dimension $\rho(d, g, r)$, and a general element of this component is base point free(and simple if $r \geq 2$).*

The proof of this theorem also gives a new proof of the former part of Theorem 2.2. In 1996, G. Martens proved that Clifford's Theorem and Theorem 2.3. become stronger for curves of odd gonalities.

Theorem 3.6. *([M1]) Let C be a curve of genus g. Assume the gonality of C is odd. Then*

(1) dim$|D| \leq \frac{d}{3}$ *for any a special linear series* $|D|$ *of degree* d *on* C*.*

(2) dim$W^r_d(C) \geq d - 3r$ *for any* $d < g$*, and if equality occurs for some* $d \leq g-2$ *and* $r > 0$*, then* C *is either trigonal, smooth plane sextic, birational to a plane septic curve or an extemal space curve of degree 10.*

Keem and Kato recently extended this theorem, as Mumford gave an extension of H. Martens theorem.

Theorem 3.7. *([KK2]) Let* C *be a curve of genus* g*. Assume the gonality of* C *is odd and dim*$W^r_d(C) \geq d - 3r - 1$ *for some* $d \leq g-4$ *and* $r > 0$*. Then* C *is a 5-gonal with* $10 \leq g \leq 18$*,* $g = 20$ *or 7-gonal of genus 21.*

4. Family of smooth curves

Let $\mathcal{I}'_{d,g,r}$ be the union of the irreducible components of the Hilbert scheme $\mathcal{H}_{d,g,r}$ whose general points correspond to smooth nondegenerate curves of degree d and genus g in $\boldsymbol{P}^r$. If the Brill-Noether number $\rho(d,g,r) = g - (r+1)(g-d+r) > 0$, then $\mathcal{I}'_{d,g,r}$ has the unique dominating component(: The locus of smooth models of its general members is dense in the moduli space $\mathcal{M}_g$ of smooth curves of genus g.). In [H], it has been conjectured that $\mathcal{I}'_{d,g,r}$ is irreducible for $\rho(d,g,r) > 0$. And the examples of reducible $\mathcal{I}'_{d,g,r}$ with positive $\rho(d,g,r)$ have been given by the existences of other components of $\mathcal{I}'_{d,g,r}$ over the locus containing $\mathcal{M}^1_{g,k}$ of $k-$gonal curves in $\mathcal{M}_g$.(See [H], [MS].) In those cases, the degree d is of the form $2g-2-lk$ for some $l \geq 2$ and $k \geq 3$ with $d \leq A_0(g,r)$, where $A_0(g,r) = \frac{2r-4}{r+1}g + \frac{r+13}{r+1}$. But we can show Theorem 4.2 and Theorem 4.3 by the following lemma in[CKM2].

Lemma 4.1. *[CKM2] Let* C *be a general* $k-$*gonal curve of genus* g $(k \geq 2)$*,* $0 \leq m, n \in \boldsymbol{Z}$ *such that* $g \geq 2m + n(k-1)$ *and let* $D \in C^{(m)}$*. Assume there is no* $E \in g^1_k$ *with* $E \leq D$*. Then dim*$|ng^1_k + D| = n$*.*

Theorem 4.2. *([K2]) Let* $\rho(d,g,r) > 0$*. Suppose* $\frac{2g+4r+10}{3} \leq d \leq A_1(g,r) - 4 + a$*,* $r \geq 6$*, where* $A_1(g,r) = \frac{2r-6}{r+1}g + \frac{2r+26}{r+1}$ *and* $2g-2-[A_1(g,r)] \equiv a$ *(mod 4),* $1 \leq a \leq 4$*. Then* $\mathcal{I}'_{d,g,r}$ *has another irreducible component over* $\mathcal{M}^1_{g,4}$*.*

Theorem 4.3. *([K2]) Assume* $\rho(d,g,r) > 0$*. Let* $r \geq 8$ *and* $d > A_0(g,r)$ *with the assumption* $g \geq 19$ *in case* $r = 8$*. Then* $\mathcal{I}'_{d,g,r}$ *is irreducible.*

A smooth curve C in $\boldsymbol{P}^r$ is said to be projectively normal if the morphisms $H^0(\boldsymbol{P}^r, \mathcal{O}_{\boldsymbol{P}^r}(m)) \to H^0(C, \mathcal{O}_C(m))$ are surjective for every number m. We could also show the following about projectively normal embeddings of k-gonal curves by Theorem 3.4 and Lemma 4.1.

Theorem 4.4. *([K3]) Let* C *be a general* k*-gonal curve of genus* g*. And let* $|F|$ *be the unique pencil of degree* k *on* C *and* K_C *the canonical divisor on* C*. Then linear series* $|K_C - mF|$ *gives projectively normal embedding of* C *for* $0 \leq mk < \frac{g-1}{2}$*.*

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