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SOME RECENT TOPICS IN SMOOTHABLE STABLE LOG SURFACES

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Abstract. This is an introductory article of the study of smoothable stable log surfaces. First we briefly introduce the projectivity of the moduli functor of smoothable stable log surfaces with given Hilbert function. The positivity of the second Chern class of stable log surfaces and its relation of boundedness are discussed. Also the study of limits of plane curves by using smoothable stable log surfaces, and the construction of surfaces by using smoothable stable surfaces are shortly discussed.

0. Introduction

It has been known for a long time that the set of all curves of a fixed genus g naturally corresponds to the points on a certain variety with dimension $3g-3$, which is denoted by M_g. The moduli space M_g is not projective, and there has been considerable work in finding a natural compactification. What kind of curves correspond to the boundary of M_g? An answer was provided by Deligne and Mumford in [DM]. This compactification is called Deligne-Mumford compactification of M_g, denoted by $\overline{M_g}$. A stable curve of genus g is defined as a 1-dimensional projective connected curve C which has only ordinary nodes as singularities such that ω_C is ample. The projectivity of $\overline{M_g}$ over any field is accomplished via the geometric invariant theory (this theory was started by Hilbert, and it was developed by Nagata, Haboush and others) and the theory of moduli (it was mainly developed by Grothendieck) by Mumford, Gieseker and others.

Via the geometric invariant theory, Gieseker [Gi] succeeded in proving the existence of a coarse moduli space $M_{K^2,\chi}$ for canonically embedded smooth projective surfaces of general type with fixed numerical invariants K^2 and χ. The next natural problem is how to construct a geometric compactification of moduli of surfaces via similar way as $\overline{M_g}$: With the proof of the boundedness conjecture in dimension two, Alexeev [A1] finished the construction of projective coarse moduli space of surfaces of general type with fixed K^2 that was started in [KSB]. The compactified moduli space should include (possibly reducible) surfaces with ordinary double curves and certain other mild singularities. These surfaces are called smoothable stable surfaces. This notion of smoothable stable surface can be generalized to smoothable stable log surface similar as the generalization of stable curve to stable n-pointed curve.

In section 1, we introduce briefly a stable log variety and the projectivity of the moduli functor of smoothable stable log surfaces with given Hilbert function. In section 2, the positivity of the second Chern class of stable log surfaces and

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its relation of boundedness are discussed. And in section 3, we briefly introduce the recent work of Hassett [Ha] for smoothable stable log surfaces and limits of plane curves. In section 4, the construction of surfaces by using smoothable stable surfaces is shortly discussed. We work throughout over the complex number field $\mathbb{C}$ by the simplicity. The notation here follows the standard text [Hartshorne's algebraic geometry].

1. Stable log variety

We recall what a stable n-pointed curve is,

Definition 1.1. A stable n-pointed curve is a collection $(C\,; P_1, \cdots P_n)$, where

1. C is a connected projective curve and $P_1, \cdots P_n$ are distinct points on C,
2. (condition on singularities) C is reduced and has nodes only, and $P_1, \cdots P_n$ lie in the nonsingular locus of C,
3. (numerical condition) for every smooth rational curve $E \subset C$, E has at least 3 special points (one of P_i or the nodes), and for every smooth elliptic curve or a rational curve with one node $E \subset C$, E has at least one special point, i.e. $K_C + \sum_{i=1}^n P_i$ is ample.

A stable n-pointed curve over a scheme S is a flat projective morphism

$$\pi\colon (C\,; P_1, \cdots P_n) \to S$$

such that $P_i \subset C$ is closed subschemes and each $P_i \to S$ is also flat, whose every fiber is a stable n-pointed curve. The moduli stack of stable n-pointed curves is coarsely represented by a projective scheme $\overline{M}_{g,n}$, see [Kn], [Mu] for $\overline{M}_{g,0}$.

Question. What is the analogue of this in higher dimensions?

The notion of nodes on a curve can be changed to semi log canonical singularities in higher dimensions.

Definition 1.2. (cf. [K et], [KM]) We say that a log variety (X, B) has semi log canonical singularities if

1. X satisfies Serre's condition S_2,
2. X has normal crossing singularities in codimension one,
3. $K_X + B$ is $\mathbb{Q}$-Cartier, and for any birational morphism $f\colon Y \to X$ from a normal $\mathbb{Q}$-Gorenstein variety Y we have
$$K_Y \equiv f^*(K_X + B) + \sum a_i E_i$$
where all $a_i \geq -1$.

Definition 1.3. A n-dimensional stable log variety is the pair (X, B) where

1. X is a n-dimensional connected projective variety and B is a reduced divisor on X,
2. (condition on singularities) the pair (X, B) has semi log canonical singularities,
3. (numerical condition) $K_X + B$ is ample.

It is well known that a stable log variety (X, B) has finite automorphism groups by [I]. A stable log variety $(X, B; L)$ over a scheme S of level N is a flat projective morphism $\pi\colon (X, B; L) \to S$ with $B \subset X$ closed schemes, $B \to S$ is also flat, and L is an invertible sheaf on X, where every fiber is a stable log variety such that the restriction of L on each fiber coincides with $\mathcal{O}(N(K_X + B))$.

Conjecture 1.4. (boundedness conjecture) For every positive rational number A, there exist

1. a positive integer N with the property that for every stable log variety (X, B) with $(K_X + B)^n = A$, the sheaf $\mathcal{O}(N(K_X + B))$ is invertible,
2. there exist a flat family $(X, B; L) \to S$ over a scheme of finite type whose fibers include all stable log varieties of level N with $(K_X + B)^n = A$.

This boundedness conjecture has been proved in dimension two [A1] and clearly in dimension one. With this proof Alexeev finished the construction of projective coarse moduli space of surfaces of general type with fixed K^2 that was started in [KSB]. Define the functor

$$\begin{aligned}\overline{M}_A^N(S) = &\{n - dimensional\ \ stable\ \ log\ \ variety\ \ over\ \ S\\ &of\ \ level\ \ N\ \ with\ \ (K_X + B)^n = A\}/\simeq\end{aligned}$$

We say that two pairs $(X_1, B_1; L_1)$ and $(X_2, B_2; L_2)$ are isomorphic if there exists an isomorphism of (X_1, B_1) and (X_2, B_2) over S that induces a fiber-wise isomorphism of L_1 and L_2. In [A2], he shows that the existence of coarse moduli space for stable log varieties under the assumption of log MMP$(n+1)$ together with boundedness conjecture for dimension n.

Every stable curve is smoothable. But if $n \geq 2$, there is a small submoduli space of smoothable stable log varieties inside of a moduli space of stable log varieties. Smoothability means that (X, B) admits a deformation $(\mathcal{X}, \mathcal{B})$ over Spec of a discrete valuation ring where generic fiber X_t has rational Gorenstein singularities together with $K_{X_t} + B_t$ Cartier. Furthermore $K_{\mathcal{X}} + \mathcal{B}$ is $\mathbb{Q}$-Cartier. Smoothable stable varieties ($B = \emptyset$) are added to the boundary points in the compactified the moduli space of smooth n-folds of general type. The moduli functor $\overline{M}_H^{sm}$ assigns to a scheme S the set of isomorphism classes of families of stable smoothable n-folds over S with a given Hilbert function $H(\ell) = \chi(X, \mathcal{O}(\ell K_X))$.

Theorem 1.5. [Ka] Under the assumption of MMP$(n+1)$, the moduli functor $\overline{M}_H^{sm}$ is coarsely represented by a projective scheme.

Example 1.6. [Bl] Take lines $L_1, \cdots L_r \in \mathbb{P}^2 (r \geq 7)$ in general position ($L_i \cap L_j \cap L_k = \emptyset$ for any i, j, k which are pairwise different). Let $L = \cup L_i$, $S = Sing\, L$ and $\pi\colon B_S\mathbb{P}^2 \to \mathbb{P}^2$ the blow up of each point of S. Then $\bar{L}_i^2 = 1 - (r-1) = 2 - r \leq -2$ where $\bar{L}_i$ is the proper transform of L_i. Let $\sigma\colon B_S\mathbb{P}^2 \to X_r$ denote the contraction of $\bar{L}_i$ for all i. Then X_r has r-number of rational singularities and K_{X_r} is ample. We have

$$K_{X_r}^2 = \frac{1}{2}(r-6)^2\frac{r-1}{r-2}, \quad \chi(\mathcal{O}_{X_r}) = 1.$$

It is clearly that X_r is a stable surface but it is not smoothable stable surface ($K_{X_r}^2$ is not integer). In the smooth case there is only a finite number of families of minimal surfaces X of general type with $\chi(\mathcal{O}_X) = 1$; this is obtained by the quasi-projectivity of $M_{K^2,\chi}$ [Gi] and the Bogomolov-Miyaoka-Yau inequality $K^2 \leq 9\chi$. It is also true for smoothable stable surfaces by Theorem 1.5. But it is not true for stable surfaces by this example.

2. Positivity results for stable log surfaces

If X is a minimal surface of general type then Bogomolov-Miyaoka-Yau inequality implies that $c_2(X) \geq \frac{1}{3}K_X^2$. This can be generalized to stable log surfaces. We first need to introduce some definitions of singularities.

Definition 2.1. Let X be a normal variety and $D = \sum d_i D_i$ an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f\colon Y \to X$ be a proper birational morphism. We write

$$K_Y \equiv f^*(K_X + D) + \sum a_i E_i.$$

We define

$$discrep(X, D) = \inf_{E_i}\{a_i | E_i \text{ is exceptional, and } Center_X(E_i) \neq \emptyset\}$$

We say that (X, D), or $K_X + D$ is

$$\left.\begin{array}{l}\text{terminal}\\ \text{canonical}\\ \text{purely log terminal}\\ \text{log canonical}\end{array}\right\} \quad \text{if} \quad discrep(X, D) \quad \left\{\begin{array}{l}> 0,\\ \geq 0,\\ > -1,\\ \geq -1.\end{array}\right.$$

Moreover, (X, D) is *Kawamata log terminal* (klt) if (X, D) is purely log terminal, $d_i < 1$ for every i; and (X, D) is *divisorial log terminal* (dlt) if there exists a log resolution so that exceptional locus consists of divisors with all $a_i > -1$.

Let (Y, D) be a dlt pair such that D is a reduced Weil divisor. The second Chern class of a dlt pair can be defined as orbifold Euler number (cf. [Mi], [Wa] or [Me]). Define $Sing(Y, D)^I$ to be the set of singular points of Y which lie outside of D. Then

$$c_2(Y, D) = e_{top}(Y) - e_{top}(D) - \sum_{p \in Sing(Y,D)^I} (1 - \frac{1}{r(p)})$$

where $r(p)$ is the local orbifold fundamental group. And $c_1(Y, D) = K_Y + D$.

Let (X, B) be a stable surface. Then each component of the normalization of (X, B) has log canonical singularities. Assume that X is normal. Then by the well-known theorem of dlt model of a log canonical pair of surfaces, the proof can be obtained by the classification of log canonical surfaces, there exists a projective birational morphism $f\colon Y \to X$ with (Y, D) a dlt pair such that $K_Y + D = f^*(K_X + B)$ and D is reduced and $c_2(X, B) = c_2(Y, D)$.

Theorem 2.2. (cf. [Mi], [Wa] or [Me]) Let (X, B) be a log canonical pair such that $K_X + B$ is nef and big. Then $c_2(X, B) \geq \frac{1}{3}(K_X + B)^2$. In particular $c_2(X, B) > 0$ if $K_X + B$ is ample.

Theorem 2.3. [L1] Let X be a smoothable stable surface. Then the number of components in X is bounded by $12\chi(\mathcal{O}_X) - K_X^2$.

In fact, the number of singular points of X which lie outside of B is also bounded. But the number of singular points on double curves is not necessary to be bounded.

Example 2.4. Consider the following central fiber X of a degeneration of $K3$ surfaces with involutions. Let $X = X_1 \cup \cdots \cup X_n$, where X_1, X_n are rational surfaces and $X_i = \mathbb{P}^1 \times E$ (elliptic curve) if $2 \le i \le n-1$. This gives a degeneration of Enrique surfaces with a central fiber having four nodal singular points on each double curve. Therefore there are $4(n-1)$ singular points on the double curves.

Let (X, B) be a smoothable stable log surface. Then by 3-dimensional log MMP there exists a semi-dlt (Y, D) (each pair of a component Y_i, and $D|_{Y_i}$ plus a double curve is dlt) such that $c_2(Y, D) = \sum c_2(Y_i, D_i + F_i)$ where F_i a double curve. By same method in [L1], we obtain the following:

Theorem 2.5. Let (X, B) be a smoothable stable log surface and let $(\mathcal{X}, \mathcal{B})$ be a deformation. Then the number of components in (X, B) is bounded by the second Chern class of generic fiber $c_2(X_t, B_t)$.

Example 2.6. Let $X = C \times C$ such that C is a union of two smooth elliptic curves E_1, E_2 with $E_1 \cap E_2 = \{p\}$. Then X is a smoothable stable surface with $12\chi(\mathcal{O}_X) - K_X^2 = 4$. Therefore the inequality in Theorem 2.3 is sharp.

Let $Z = E \times E$ such that E is a smooth elliptic curve and p_1, p_2 are two natural projections. Let $B = p_1^*(p) + p_2^*(q)$ where p, q are points in E. Then (Z, B) is a smoothable stable log surface with $c_2(Z, B) = 1$. So the inequality in Theorem 2.5 is also sharp.

3. Smoothable stable log surfaces and limits of plane curves

Let P_d be a coarse moduli space for smooth plane curves of degree $d \ge 4$. For each $C \in P_d$, $(\mathbb{P}^2, C)$ is a stable log surface. Let $\overline{P}_d$ denote the closure of P_d in the moduli space of smoothable stable log surfaces. We have a morphism $j\colon P_d \to M_{g(d)}$ where $M_{g(d)}$ is the moduli space of curves of genus $g(d) = \frac{1}{2}(d-1)(d-2)$. This morphism j extends to $\bar{j}\colon \overline{P}_d \to \overline{M}_{g(d)}$ because B has only nodal singularities for $(X, B) \in \overline{P}_d$ and $K_X + B|_B = K_B$ is ample. Since j is isomorphic onto its image, $\bar{j}$ is birational onto its image.

Question. Is $\bar{j}\colon \overline{P}_d \to \overline{M}_{g(d)}$ isomorphic onto its image? Does $\bar{j}$ ever have positive dimension fibers?

Theorem 3.1. [Ha] $\bar{j}\colon \overline{P}_4 \to \overline{M}_3$ is an isomorphism.

Example 3.2. [Ha] Consider the map $\bar{j}\colon \overline{P}_5 \to \overline{M}_6$. Every hyperelliptic curves C of genus 6 is contained in the closure of the plane quintic curves (cf. [H], [Gr]). The limiting g_5^2 takes the form $2g_2^1 + p$ where p is any one of the 14 Weierstrass points of C. Each of these Weierstrass points yields an element at the boundary of $\overline{P}_5$. The construction is the following: Let C be a hyperelliptic curve of genus 6 with a double cover $r\colon C \to \mathbb{P}^1$. We have $r_*\mathcal{O}_C = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-7)$, so that we can regard C as a bisection of the rational ruled surface $\mathbb{F}_7$. C is disjoint from the zero section $C_0 (C_0^2 = -7)$. Let f be a ruling of $\mathbb{F}_7$, tangent to C at p. Let $\pi\colon P \to \mathbb{F}_7$ be obtained by taking the minimal embedded resolution of $C \cup f$, and then blowing up at the intersection point of the proper transform of C with the exceptional locus. Let X be obtained by contracting two exceptional -2-curves, the proper transform of f, and the proper transform of C_0. Let B be the image of proper transform of C in X. Then (X, B) is a smoothable stable log surface [Ma] (smoothable to $(\mathbb{P}^2, C)$ where C is a plane quintic curve). X has one quotient singularity outside of B

whose minimal resolution graph is: $\overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-7}{\circ}$, it is clearly a singularity of class T [KSB].

Let (X, B) be a stable log surface. Assume that B is Cartier. Then we have two long exact sequences for deformation of pairs, which generalize the long exact sequences in a smooth case (cf. [Ha]).

1. $0 \to \mathrm{T}^0_{X,B} \to \mathrm{T}^0_X \to H^0(B, \mathcal{O}_B(B)) \to \mathrm{T}^1_{X,B} \to \mathrm{T}^1_X \to H^1(B, \mathcal{O}_B(B)) \to \mathrm{T}^2_{X,B} \to \mathrm{T}^2_X$ where $\mathrm{T}^i_X = \mathrm{Ext}^i_X(\Omega^1_X, \mathcal{O}_X)$ for $i = 0, 1, 2$,
2. $0 \to \mathrm{T}^0_X(-B) \to \mathrm{T}^0_{X,B} \to \mathrm{T}^0_B \to \mathrm{T}^1_X(-B) \to \mathrm{T}^1_{X,B} \to \mathrm{T}^1_B \to \mathrm{T}^2_X(-B) \to \mathrm{T}^2_{X,B}$ where $\mathrm{T}^i_X(-B) = \mathrm{Ext}^i_X(\Omega^1_X, \mathcal{O}_X(-B))$ for $i = 0, 1, 2$.

The stability implies that $\mathrm{T}^0_{X,B} = 0$ and $T^0_B = 0$.

Let (X, B) be a smoothable stable log surface. Assume that B is Cartier. Consider the sheaves $\mathcal{T}^i_X = \mathcal{E}xt^i_X(\Omega^1_X, \mathcal{O}_X)$ for $i = 0, 1, 2$. Define $\tilde{\mathcal{T}}^i_X \subset \mathcal{T}^i_X$ which represents the infinitesimal smoothable deformations: Let $U \subset X$ be an analytic neighborhood with index one cover V. Define $\tilde{\mathcal{T}}^i_X(U)$ as the invariant part of $\mathcal{T}^i_V$. By this definition, it is obvious that $\tilde{\mathcal{T}}^i_X \subset \mathcal{T}^i_X$ and $\tilde{\mathcal{T}}^0_X = \mathcal{T}^0_X$.

Define $\tilde{\mathrm{T}}^0_X = \mathrm{T}^0_X$, and $\tilde{\mathrm{T}}^1_X$ as the elements of T^1_X mapped to $H^0(\tilde{\mathcal{T}}^1_X)$. And the obstruction map can be computed by $\tilde{\mathrm{T}}^1_X \subset \mathrm{T}^1_X \xrightarrow{ob} \mathrm{T}^2_X$. Define $\tilde{\mathrm{T}}^0_{X,B} = \mathrm{T}^0_{X,B}$, and $\tilde{\mathrm{T}}^1_{X,B}$ as the elements of $\mathrm{T}^1_{X,B}$ mapped to $\tilde{\mathrm{T}}^1_X$. Also define $\tilde{\mathrm{T}}^0_X(-B) = \mathrm{T}^0_X(-B)$, and $\tilde{\mathrm{T}}^1_X(-B)$ as the elements of $\mathrm{T}^1_X(-B)$ mapped to $H^0(\tilde{\mathcal{T}}^1_X)$. Then we have two long exact sequences (cf. [Ha]) for a smoothable stable log surface (X, B),

1. $0 \to \tilde{\mathrm{T}}^0_X \to H^0(B, \mathcal{O}_B(B)) \to \tilde{\mathrm{T}}^1_{X,B} \to \tilde{\mathrm{T}}^1_X \to H^1(B, \mathcal{O}_B(B))$,
2. $0 \to \tilde{\mathrm{T}}^1_X(-B) \to \tilde{\mathrm{T}}^1_{X,B} \to \mathrm{T}^1_B$.

Furthermore we assume that X is Gorenstein along B. Then $\tilde{\mathcal{T}}^1_X$ and $\mathcal{T}^1_X$ coincide along B and $\tilde{\mathcal{T}}^1_X(-B) = \tilde{\mathcal{T}}^1_X \cap \mathcal{T}^1_X(-B)$. And $\tilde{\mathrm{T}}^1_X(-B)$ can be computed as the local-global spectral sequence,

$$0 \to H^1(\mathcal{T}^0_X(-B)) \to \tilde{\mathrm{T}}^1_X(-B) \to \ker[H^0(\tilde{\mathcal{T}}^1_X(-B)) \to H^2(\mathcal{T}^0_X(-B))] \to 0.$$

Theorem 3.3. [Ha] Let (X, B) be a smoothable stable log surface such that B is Cartier and X is Gorenstein along B. Assume that $H^1(\mathcal{T}^0_X(-B)) = 0$ and $H^0(\tilde{\mathcal{T}}^1_X(-B))$ injects into $H^2(\mathcal{T}^0_X(-B))$. Then $\tilde{\mathrm{T}}^1_{X,B}$ injects into T^1_B.

Corollary 3.4. [Ha] Let C be a stable plane curve with degree $d \geq 4$. Then $\overline{P}_d$ is smooth (as a stack) at $(\mathbb{P}^2, C)$ and the derivative of $\bar{j}$ is injective.

Question. Are the next three steps true in the proof of $H^0(\tilde{\mathcal{T}}^1_X(-B)) = 0$ for $(X, B) \in \bar{P}_d$?

1. The support of sheaf $\tilde{\mathcal{T}}^1_X$ lies in the union of double curves,
2. every component in double curves is $\mathbb{P}^1$,
3. $H^0(\tilde{\mathcal{T}}^1_X(-B)) = \sum H^0(\mathcal{O}_{\mathbb{P}^1}(-a))$ for $a < 0$.

The first step may be obtained by the generalization of global smoothings in [Fr], and the second step may be proved by the generalization of Hodge theory argument (the singularities of $(X, B) \in \bar{P}_d$ should be milder than general case). But we have no reason for the third one if $(K_X + B)^2$ increases.

4. Construction of surfaces by using smoothable stable log surfaces

Recently Craighero, Gattazzo, Dolgachev and Werner [DW] construct a new simply connected Godeaux surface, a minimal surface of general type with $p_g = q = 0, K = 1$, (Barlow [Ba] constructed the first example of simply connected Godeaux surface) by the minimal resolution of a quintic surface with 4 elliptic singularities (locally $x^2 + y^3 + z^6 = 0$). It is often possible to construct a new type of surface via the resolution of singularities or smoothings of a known singular surface.

Question. Is there a Dolgachev surface Y (a minimal elliptic surface with $p_g = g = 0$, $P_2 = 1$, and $\kappa = 1$) containing $D = \mathbb{P}^1$ with $D^2 = -4$ such that the natural map $H^1(T_Y) \to H^1(N_{D|Y})$ is surjective?

If this question is true then we can construct a Godeaux surface by using a smoothable stable surface X where X is obtained by the contraction of D in Y. The smoothability is proved in [L2].

Example 4.1. Let X be a Godeaux surface with an involution. Assume that the fixed divisor is the -3-curve D ($D = \mathbb{P}^1$ with $D^2 = -3$). Furthermore, assume that $H^2(T_X) = 0$ and the natural map $H^1(T_X) \to H^1(N_{D|X})$ is surjective. The fixed locus of a Godeaux surface with an involution is classified in [KL]. There is an example which satisfies the above conditions [L2]. Let Y be a blowing up at a point in D. Let $\bar{D}$ be the proper transform of D in Y. And let Z be a singular surface obtained by the contraction of $\bar{D}$ ($\bar{D}^2 = -4$). The smoothablity of Z is proved in [L2]. But Z is not stable surface because $K_Z\bar{E} < 0$ where $\bar{E}$ is the image of -1-curve. Let $\mathcal{X} \to \operatorname{Spec} R$ (R is a discrete valuation ring) be a smoothings of Z. Then the canonical model of $\mathcal{X} \to \operatorname{Spec} R$ is obtained by flip [L2]. In fact, the canonical model is a deformation of X which makes to disappear -3-curve in X.

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