Trends in Mathematics
Information Center for Mathematical Sciences
Volume 2, December 1999, Pages 86–90

ON THE ZEROES OF HOLOMORPHIC VECTOR FIELDS ON ALGEBRAIC MANIFOLDS

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Given a holomorphic vector field V on a complex algebraic manifold X, the problems we will survey here are about the relation between the geometry of X and the geometry of zeroes of V. Here, we will assume that the zero set $Z(V)$ of V is non-empty. The integral curves of V define a foliation of $X - Z(V)$ by analytic curves. In general, these analytic curves are non-algebraic. However, most problems can be reduced to the case when the integral curves are all algebraic. As a matter of fact, most of the problems can be reduced to the case when V is induced by an algebraic action of the additive algebraic group $G_a(\mathbf{C})$ on X for the following reason.

When X is a projective manifold, or more generally when X is a Moishezon manifold, i.e., a non-singular algebraic space, it is well-known that the identity component $Aut_o(X)$ of the group of automorphisms of X is an algebraic group. The 1-parameter subgroup $exp(tV)$ of automorphisms of X induced by the vector field V is a 1-dimensional commutative analytic subgroup of $Aut_o(X)$. Let G_V be the Zariski closure of $exp(tV)$ in $Aut_o(X)$. Then G_V is an algebraic subgroup of $Aut_o(X)$, which is commutative because it contains a Zariski dense commutative subgroup $exp(tV)$. Pick a point $z \in Z(V)$. Then z is a fixed point under the action of G_V on X, because it is fixed by the Zariski dense subgroup $exp(tV)$. Consider the action of G_V on the jet spaces at z. Its action on the jet space of sufficiently higer order must be faithful, because an algebraic group action which is trivial in all jets at z must be trivial on the whole X. Thus G_V has a faithful linear representation on some high order jet space at z. It follows that G_V is a commutative linear algebraic group. By the structure theory of commutative linear algebraic group, G_V is of the form $(G_a(\mathbf{C}))^k \times (G_m(\mathbf{C}))^l$ for some integers k, l, where $G_a(\mathbf{C})$ and $G_m(\mathbf{C})$ denote the additive group and the multiplicative group over the complex numbers. By Bialynicki-Birula decomposition ([BB]), the structure of the fixed point sets and orbits of $G_m(\mathbf{C})$-action on algebraic manifolds are very well-understood. Thus the difficult part is to understand when $G_V = G_a(\mathbf{C})$. For convenience, we will say that V is additive if $G_V = G_a(\mathbf{C})$ and multiplicative if $G_V = G_m(\mathbf{C})$.

1. Vector fields vanishing on a hypersurface

The first circle of results that we survey here is the case when a component of $Z(V)$ is a hypersurface in X. The main question is what we can say about X in this case. Of course, given such V and X, we can consider the product $X \times Y$ with an arbitrary projective manifold Y and the naturally induced vector field V' on

1991 *Mathematics Subject Classification.* 14J45, 14J50.
Key words and phrases. vector fields, rational varieties, Fano manifolds.

$X \times Y$ so that $Z(V')$ has a codimension 1 component on $X \times Y$. Thus to have an interesting result, we must exclude the case when X and $Z(V)$ could come from a product construction.

One condition that exclude this possibility is when the codimension 1 component of $Z(V)$ is an ample hypersurface. Let L be the ample line bundle corresponding to that hypersurface. By looking at the action of G_V on the graded ring $\oplus_{k=o}^{\infty} H^0(X, \mathcal{O}(kL))$, Mori and Sumihiro proved the following:

Theorem [**MS**] *Let X be a projective manifold which has a non-zero holomorphic vector field vanishing on an ample hypersurface D. Then $X \cong \mathbf{P}_n$ and D corresponds to a hyperplane.*

This result was sharpened to the following form by Wahl.

Theorem [**Wa**] *Let L be an ample line bundle on a projective manifold X. If $H^0(X, \Theta_X(-L))$ has a non-zero section, then $X \cong \mathbf{P}_n$ and $L \cong \mathcal{O}(1)$.*

The difference from Mori-Sumihiro case is that L is not assumed to have a non-trivial section. Note that the zero section of the negative line bundle L^{-1} can be blown-down to get an algebraic variety L' with an isolated singularity. Wahl's idea is to show that L' is in fact non-singular by using a vector field on L' induced by a non-zero section of $\Theta_X(-L)$.

Another condition that exclude the possibility of X being a product of two manifolds is that the Picard number of X is 1. So assume that the Picard number of X is 1 and there exists a vector field V vanishing on a hypersurface D. If X is projective, then D is ample and we are reduced to the case of Mori-Sumihiro. But if X is just an algebraic space, then the ampleness of D is unclear and Mori-Sumihiro cannot be applied. This is the case treated in [Hw2].

Theorem [**Hw2**] *Let X be a non-singular algebraic space, i.e., a Moishezon manifold, which has Picard number 1. Suppose there exists a vector field vanishing on a hypersurface D. Then $X \cong \mathbf{P}_n$ and D corresponds to a hyperplane.*

The key ingredient of the proof is a structure theorem for additive vector fields vanishing on hypersurfaces which was proved in [Hw1] using the Poincare normal form theory of autonomous systems. An immediate application of above theorem is a simple proof of the following result of Siu, which answered a problem raised by Kodaira-Spencer in [KS].

Theorem [**Si**] *Let $\pi : \mathcal{X} \to \Delta := \{t \in \mathbf{C}, |t| < 1\}$ be a smooth proper morphism from a complex manifold $\mathcal{X}$ to the unit disc. If a fiber of π is $\mathbf{P}_n$, then any other fiber of π is $\mathbf{P}_n$.*

By $H^1(\mathbf{P}_n, \Theta) = 0$, we see that a generic fiber of π is $\mathbf{P}_n$ and the problem is what happens to the spicial fibers. Siu's original proof used several vector fields to analyze the orbit structure. A simpler proof is using the above result [Hw2]: just observe that any special fiber is Moishezon and has vector fields vanishing on a hypersurface by upper semi-continuity. The ideas of Siu's original proof are used in genealizing the above result to hyperquadrics:

Theorem [**Hw1**] *Let $\pi : \mathcal{X} \to \Delta := \{t \in \mathbf{C}, |t| < 1\}$ be a smooth proper morphism from a complex manifold $\mathcal{X}$ to the unit disc. If a fiber of π is a smooth hyperquadric, then any other fiber of π is a smooth hyperquadric.*

The essential part of the proof is to apply the Poincare normal form theory to exterior products of vector fields. Here individual vector field does not vanish on a hypersurface but we exploit the fact that the exterior product of some of the vector fields involved vanishes on a hypersurface to a higher order.

It should be mentioned that the above two Theorems [Si], [Hw1] are immediate if we assume that π is projective from the works of Hirzebruch-Kodaira and Brieskorn ([HK] [Br]).

2. Vector fields vanishing at isolated points

The opposite of the case when V vanishes on a hypersurface is when $Z(V)$ consists of finitely many points. The major problem here is the following conjecture of J. Carrell, which first appeared in the early 1970's:

Carrell's Conjecture *Let X be a projective manifold with a vector field V whose zero set consists of finitely many points. Then X is rational.*

¿From the classification of surfaces, one can easily check that this conjecture is true for surfaces. It is true when the vector field V is assumed to be multiplicative by Bialynicki-Birula theory ([BB]). If $Aut_o(X)$ is assumed to be a reductive group, then one can produce a multiplicative vector field from an additive vector field by Jacobson-Morozov lemma, and the conjecture is true.

Let us review some partial results on the conjecture here. First of all, for low dimensions, we proved the following.

Theorem [Hw3] *When* $\dim(X) \leq 5$, *Carell's conjecture is true.*

Here the essential idea is to introduce a blow-up procedure to reduce the problem to the case when the additive vector field vanishes on a hypersurface, and then apply the result from [Hw1]. A similar technique works in arbitray dimension, if one assumes that the linear part of the vector field at the isolated zero is identically zero:

Theorem [Hw4] *In Carrell's conjecture, assume that the linear part of V at a zero point vanishes identically. Then X is rational.*

Another special case of the conjecture is the following result of Konarski, which is in a sense the opposite to the case studied above in [Hw4]:

Theorem [Kn] *In Carrell's conjecture, assume that the linear part of V at a zero point has a single Jordan block. Then X is rational.*

Konarski's idea was to deform the vector field so that we can apply the theory of multiplicative vector fields.

3. Algebraicity question

Suppose X is a compact Kähler manifold which has a vector field V. Carrell and Lieberman showed that if $\dim(Z(V)) \leq 1$, then V is projective algebraic. This is an immediate corollary of the following vanishing theorem of theirs:

Theorem [CL] *Let X be a compact Kähler manifold with a holomorphic vector field V whose zero set has dimension $< k$. Then $H^p(X, \Omega^q) = 0$ for $|p-q| \geq k$.*

The idea of the proof is that the contraction of differential forms by the vector field V serves as a kind of homotopy operator for the Hodge-de Rham spectral sequence outside the zero set of V. For the special case when $Z(V)$ has dimension 0, a simplified proof is given in [GH] pp. 708-712.

Note that the assumption $\dim(Z(V)) \leq 1$ implies that the subvariety $Z(V)$ is projective algebraic. Thus the next result is a generalization of Carrell-Lieberman's result:

Theorem [Hw3] *Let X be a compact Kähler manifold with a holomorphic vector field V. If the zero set $Z(V)$ is an algebraic variety, then X is projective algebraic.*

The idea is to blow-up certain subvarieties of $Z(V)$, to reduce it to the case when $Z(V)$ has a codimension-1 component and then use the universal Chow family of the orbital curves of the vector field.

4. Vanishing orders of vector fields

Given a complex manifold X and a holomorphic vector field V, we define the vanishing order of V at a given point $x \in X$ as follows. Choosing a local holomorphic coordinate system $(z_1, \ldots, z_n)$ at x, we can write V locally as

$$V = f_1(z)\frac{\partial}{\partial z_1} + \cdots + f_n(z)\frac{\partial}{\partial z_n}$$

where f_i's are local holomorphic function at x. Let $\mathbf{m}$ be the maximal ideal of the local ring $\mathcal{O}_x$ of germs of holomorphic functions at x. We say that the vanishing order of V at x is k if $f_i \in \mathbf{m}^k$ for each i and $f_j \notin \mathbf{m}^{k+1}$ for some j.

A natural question to ask is how large the vanishing order can be. For example, one may ask: is the vanishing order of vector fields at a given point on an n-dimensional projective manifold bounded by a number depending only on n? The answer is no. Suppose the vanishing orders at a given point $x \in X$ of all vector fields on X are bounded by a number N. Then we get a bound on the dimension of $Aut(X)$ in terms of N, by considering the Taylor expansion of vector fields at x. It is well-known that the Hirzebruch surface $\mathbf{P}(\mathcal{O}(m) \oplus \mathcal{O})$ has automorphism group of dimension $m+5$ for $m > 0$. Thus for any integer M, we can find a Hirzebruch surface X so that at a generic point $x \in X$, the vanishing order of some vector field of X is larger than M.

Since these automorphisms of Hirzebruch surfaces arise from the twisting of the fiber bundle structure, one may ask whether we can get a nice bound if the manifolds do not have fibration structure. To guarantee this, we will consider Fano manifolds of Picard number 1 here. In this case, we get the following result:

Theorem [Hw5] *Let X be an n-dimensional Fano manifold of Picard number 1 and $x \in X$ be a generic point. Then for any vector field V on X, its vanishing order at x cannot exceed n. In particular, the dimension of $Aut(X)$ is bounded by $n \times \binom{2n}{n}$.*

By Mori's bend and break trick ([Kl]), there exists a rational curve on X whose normal bundle is of the form $[\mathcal{O}(1)]^p \oplus \mathcal{O}^{n-1-p}$ for some integer $p \geq 0$. Such a rational curve is called a standard rational curve. The main idea of the proof is to give a bound on the difference of the vanishing orders at two different points on a standard rational curve. Starting from a given point x, consider a connected curve with n irreducible components each of which is a standard rational curve. These connected curves cover a Zariski dense subset of X from the Picard number condition on X. Now if the vanishing order of a vector field V at x is too big, then its vanishing orders at points of these connected curves remain big by the bound on the difference of the vanishing orders. This forces V to vanish identically on X.

It is unlikely that the above Theorem [Hw5] is sharp. In this regard, we raise the following question.

Question *Let X be a Fano manifold of Picard number 1 and $x \in X$ be a generic point of X. Is the vanishing order at x of any non-zero holomorphic vector field on X less than or equal to 2?*

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