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TORIC VARIETIES AND THE FLAG F-VECTORS OF ZONOTOPES

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Abstract. A toric variety is defined by a certain collection of cones. ¿From this fact, the properties of a toric variety have strong connection with the combinatorial structure of the corresponding fan and the relations among the generators. Toric varieties obtained from hyperplane arrangements are related to zonotopes. We investigate some properties of toric varieties, and the number of faces of zonotopes.

1. Introduction

Toric variety is a normal algebraic variety containing algebraic torus T_N as an open dense subset with an algebraic action of T_N which is an extension of the group law of T_N. A toric variety can be described in terms of a certain collection, which is called a fan, of cones. ¿From this fact, the properties of a toric variety have strong connection with the combinatorial structure of the corresponding fan and the relations among the generators. That is, we can translate the difficult algebro-geometric properties of toric varieties into very simple properties about the combinatorics of cones in affine spaces over the reals. For example, there are some sufficient or necessary conditions about projectivity for toric varieties, which are stated in terms of cones and the relations of their generators(cf. [10] and [12]). The Chow ring of toric varieties is also defined by the combinatorial structure of cones(cf. [3] and [14]) .

If there are some hyperplanes which have the zero as the only common point, and are defined over $\mathbf{Q}$, then we obtain a toric variety from this hyperplane arrangement(cf. [8] and [16]). It is known that the corresponding polytope becomes a zonotope. In this paper, we state some properties of a toric variety obtained from hyperplane arrangement, and then define the Chow ring for a toric variety. It has strong connection with the structure of cones and the number of cones in the corresponding fan. In the last part of this paper, we consider the number of faces of polytopes, and especially zonotopes.

2. Definitions and some results

Now we introduce some basic definitions which are used throughout this paper. Let N be a free $\mathbf{Z}$-module of rank r over the ring $\mathbf{Z}$ of integers, and denote by $M := \mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$ its dual $\mathbf{Z}$-module with the canonical bilinear pairing $\langle\ ,\ \rangle$:

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$M \times N \longrightarrow \mathbf{Z}$. We denote the scalar extensions of N and M to the field $\mathbf{R}$ of real numbers by $N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R}$ and $M_{\mathbf{R}} := M \otimes_{\mathbf{Z}} \mathbf{R}$, respectively.

A subset σ of $N_{\mathbf{R}}$ is called a *rational convex polyhedral cone* (or a *cone,* for short), if there exist a finite number of elements $n_1, n_2, ..., n_s$ in N such that

$$\begin{aligned}\sigma &= \mathbf{R}_{\geq 0} n_1 + \mathbf{R}_{\geq 0} n_2 + ... + \mathbf{R}_{\geq 0} n_s \\ &:= \{a_1 n_1 + ... + a_s n_s \mid a_i \in \mathbf{R}, a_i \geq 0 \text{ for all } i\},\end{aligned}$$

where we denote by $\mathbf{R}_{\geq 0}$ the set of nonnegative real numbers. σ is said to be *strongly convex* if it contains no nontrivial subspace of $\mathbf{R}$, that is, $\sigma \cap (-\sigma) = \{0\}$.

A subset τ of σ is called a *face* and denoted by $\tau \prec \sigma$, if $\tau = \sigma \cap \{m_0\}^{\perp} := \{y \in \sigma \mid \ \langle m_0, y\rangle = 0\}$ for an $m_0 \in \sigma^{\vee}$, where $\sigma^{\vee} := \{x \in M_{\mathbf{R}} \mid \langle x, y\rangle \geq 0 \text{ for all } y \in \sigma\}$ is the *dual* cone of σ.

Definition A finite collection Δ of strongly convex cones in $N_{\mathbf{R}}$ is called a *fan* if it satisfies the following conditions:

(i) Every face of any $\sigma \in \Delta$ is contained in Δ.

(ii) For any $\sigma, \sigma' \in \Delta$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.

A cone σ is said to be *simplicial* if there exist $\mathbf{R}$-linearly independent elements $\{n_1, n_2, ..., n_s\}$ in N such that σ can be expressed as $\sigma = \mathbf{R}_{\geq 0} n_1 + \cdots + \mathbf{R}_{\geq 0} n_s$. We say that a fan Δ is *simplicial* if every cone $\sigma \in \Delta$ is simplicial. A fan Δ is said to be *complete* if $|\Delta| := \bigcup_{\sigma \in \Delta} \sigma = N_{\mathbf{R}}$.

If a fan Δ is given, then there exists a toric variety $X := T_N\mathrm{emb}(\Delta)$ determined by Δ over the field $\mathbf{C}$ of complex numbers. For the precise definition of toric varieties, see [3], [4] and [10]. See also [2] and [11], for the recent develoment in toric varieties.

Definition An $\mathbf{R}$-valued function h on $|\Delta|$ is called a Δ-*linear support function* if h is $\mathbf{Z}$-valued on $N \cap |\Delta|$ and if h is linear on each cone $\sigma \in \Delta$. We denote by $\mathrm{SF}(N, \Delta)$ the additive group consisting of all Δ-linear support functions.

Important algebro-geometry properties of toric varieties can be interpreted in terms of the corresponding fans. For example,

(1) The toric variety is nonsingular if and only if the fan Δ is *nonsingular*, that is, every cone $\sigma \in \Delta$ is nonsingular. We say that a cone σ is *nonsingular* if there exist a $\mathbf{Z}$-basis $\{n_1, ..., n_r\}$ of N and $s \leq r$ such that $\sigma = \mathbf{R}_{\geq 0} n_1 + \cdots + \mathbf{R}_{\geq 0} n_s$.

(2) The toric variety is compact if and only if the corresponding fan is finite and complete, where a fan Δ is said to be *complete* if $|\Delta| = N_{\mathbf{R}}$.

(3) Let $X = T_N\mathrm{emb}(\Delta)$ be a compact toric variety. Then X is projective if and only if there exists a Δ-linear support function $h \in \mathrm{SF}(\Delta)$ such that h is *strictly upper convex* with respect to Δ. In other words, for any $\sigma \in \Delta(r)$, there exists a unique $l_\sigma \in M$ such that $h(x) = \langle l_\sigma, x\rangle$ for any $x \in \sigma$, and $\langle l_\sigma, n\rangle \geq h(n)$ for any $n \in N_{\mathbf{R}}$, where equality holds if and only if $n \in \sigma$.

For a Δ-linear support function, we define

$$\Box_h := \{m \in M_{\mathbf{R}} | \langle m, n\rangle \geq h(n), \forall n \in N_{\mathbf{R}}\}.$$

Then it becomes a convex polytope in $M_{\mathbf{R}}$, and equals the convex hull of the above l_σ for all $\sigma \in \Delta(r)$. If h is strictly upper convex, then we know more conditions about $\Box_h$.

Theorem 2.1. (Oda [10]) . *Let Δ be a fan in $N_{\mathbf{R}}$. Then Δ-linear support function h becomes strictly upper convex if and only if the set $\{l_\sigma | \sigma \in \Delta(r)\}$ is just the set of vertices of $\Box_h$.*

In this case, we have the 1-1 correspondence between the set of faces of $\Box_h$ and the set of cones of Δ, which corresponds the face F of $\Box_h$ to the cone F^+ in Δ, and the cone σ to the face σ^+ of $\Box_h$. Moreover, we have $\dim F + \dim F^+ = r$ and $\dim \sigma + \dim \sigma^+ = r$.

3. Hyperplane arrangement

Let $H_1, H_2, \cdots, H_s$ be finite hyperplanes in $N_{\mathbf{R}}$ such that $H_1 \cap H_2 \cap \cdots \cap H_s = \{0\}$ and each H_i is defined over $\mathbf{Q}$. Now we define $\Delta(r)$ as the set of closure of connected components of $N_{\mathbf{R}} - \cup_{i=1}^s H_i$, and Δ as the set of faces of cones in $\Delta(r)$. We call this Δ the *fan obtained from hyperplane arrangement*. In this case, we have

$$|\Delta(r-1)| := \bigcup_{\sigma \in \Delta(r-1)} \sigma = \bigcup_{i=1}^{s} H_i$$

Note that the fan obtained from hyperplane arrangement becomes complete, hence corresponding toric variety is compact.

For the general properties of hyperplane arrangements, see [5] and [13].

For any i, $H_i^\perp$ becomes one-dimensional vector subspace of $M_{\mathbf{R}}$, so there exists a generator m_i of $M \cap H_i^\perp$, unique up to sign. Now we define $h(n) := -\sum_{i=1}^s \ |\langle m_i, n\rangle|$, then h becomes a support function of Δ, and *upper convex*(i.e., $h(n+n') \geq h(n) + h(n')$). If a fan Δ is obtained from hyperplane arrangement, the corresponding polytope $\Box_h$ becomes a Minkovski sum of finite line segment, that is,

$$\Box_h = [-m_1, m_1] + [-m_2, m_2] + \cdots + [-m_s, m_s].$$

We call this type of polytope a *zonotope*. The vectors in the generating set are called *zones* of the zonotope. For the general properties of zonotopes, see [9] and [16]. The following is an example of a zonotope.

Example Let $m_1 = (\frac{1}{2}, \frac{1}{2}), m_2 = (0, \frac{1}{2}), m_3 = (-\frac{1}{2}, 0), m_4 = (1, -1) \in \mathbf{R}^2$, and

$$\Box = [-m_1, m_1] \ + \ [-m_2, m_2] \ + \ [-m_3, m_3] \ + \ [-m_4, m_4]$$

Then $\Box$ is a zonotope, and it can be obtained from a hyperplane arrangement $\mathcal{H}$ consisting of the hyperplanes which satisfy the equations $x = 0, y = 0, x + y = 0, x - y = 0$, respectively in $\mathbf{R}^2$. The corresponding fan equals the normal fan of $\Box$, and also the face fan of the polar polytope of $\Box$.

In the above example, we can see that the face of $\Box$ is homeomorphic to some interval $[-m_i + m_i]$. This example is not a special case. In generally, a face of a zonotope is homeomorphc to a sum of some intervals which generate the zonotope.

Proposition 3.1. (Liu [7], Ziegler [16]). *Let Z be a zonotope generated by the set $V = \{m_1, m_2, \cdots, m_s\} \subset N_{\mathbf{R}}$. Then we have the following*

(1) F is a face of Z if and only if it is of the form

$$F = \sum_{i \in P} m_i - \sum_{i \in N} m_i + \sum_{i \in I} [-m_i, m_i],$$

and there exists hyperplane H of $N_{\mathbf{R}}$ such that I is the index set of the zones in V that are on H, P is the index set of zones that are on one side of H and N is the index set of zones on the other side of H.

(2) Every face of Z is sentrally symmetric with respect to its barycenter.

¿From the above fact, we have the following property of the corresponding toric varieties.

Theorem 3.2. (Maeda [8]). *Let Δ be a fan in $N_{\mathbf{R}}$ with $r \geq 3$. Then it is obtained from hyperplane arrangement if and only if it satisfies the following : For any $\tau \in \Delta(r-1)$,*

$$\bar{\Delta}(\tau) := \{(\sigma + \mathbf{R}\tau)/\mathbf{R}\tau | \tau \prec \sigma, \sigma \in \Delta\}$$

becomes centrally symmetric. Moreover, in this case, the corresponding toric varieties must be projective.

Now we state some classifications of toric varieties obtained from hyperplane arrangement.

Let Δ be a 3-dimensional fan obtained from hyperplane arrangement. By the natural projection from $\mathbf{R}^3 - \{0\}$ onto $\mathbf{P}^2(\mathbf{R})$, hyperplane passing through the origin in $\mathbf{R}^3$ corresponds to the line in the project plane. Hence (simplicial) hyperplane arrangement in $\mathbf{R}^3$ corresponds to the (simplicial) line arrangement $\mathbf{P}^2(\mathbf{R})$. Grünbaum [5] conjectured that there are 103 simplicial line arrangements in $\mathbf{P}^2(\mathbf{R})$. Under this conjecture, Maeda [8] proved that there are 30 line arrangements which have $k \leq 15$ lines, and that among those 30 arrangements, there are only 17 arrangements which determines nonsingular fans.

Example The smallest 3-dimensional nonsingular fan Δ is obtained from the following hyperplane arrangement $\mathcal{H} = \{H_1, \cdots, H_6\}$:

$$\begin{array}{lll}
H_1 & = \{(x,y,z)|\ x=0\} & \supset \{\pm(n_2), \pm(n_3), \pm(n_2+n_3)\} \\
H_2 & = \{(x,y,z)|\ y=0\} & \supset \{\pm(n_1), \pm(n_3), \pm(n_1+n_3)\} \\
H_3 & = \{(x,y,z)|\ z=0\} & \supset \{\pm(n_1), \pm(n_2), \pm(n_1-n_2)\} \\
H_4 & = \{(x,y,z)|\ x-z=0\} & \supset \{\pm(n_2), \pm(n_1+n_3), \pm(n_1-n_2+n_3)\} \\
H_5 & = \{(x,y,z)|\ -x+y=0\} & \supset \{\pm(n_3), \pm(n_1-n_2), \pm(n_1-n_2+n_3)\} \\
H_6 & = \{(x,y,z)|\ x+y-z=0\} & \supset \{\pm(n_1-n_2), \pm(n_1+n_3), \pm(n_2+n_3)\},
\end{array}$$

where $\{n_1, n_2, n_3\}$ is the $\mathbf{Z}$-basis for $\mathbf{Z}^3$, and the set in the right side is the generating sets of $\Delta(1)$ contained in each $H_i, i = 1, \cdots, 6$.

4. Chow Ring

Let Δ be a simplicial fan for $N \cong \mathbf{Z}^r$, which may not be complete. We have defined the Chow ring $A(\Delta)$ of a toric variety as the Stanley-Reisner ring $\mathrm{SR}(N, \Delta)$ of Δ modulo the linear equivalence relation, using the corresponding simplicial fan (cf. [3], [14] and [15]).

If X is an r-dimensional compact nonsingular toric variety, then we have the following exact sequences:

$$\begin{array}{ccccccccc}
0 & \leftarrow & N_{\mathbf{R}} & \leftarrow & (T_N\mathrm{Div}(X))^*_{\mathbf{R}} & \leftarrow & A^{r-1}(X)_{\mathbf{R}} \otimes_{\mathbf{R}} \left(A^r(X)_{\mathbf{R}}\right)^* & \leftarrow & 0 \\
0 & \rightarrow & M_{\mathbf{R}} & \rightarrow & T_N\mathrm{Div}(X)_{\mathbf{R}} & \rightarrow & A^1(X)_{\mathbf{R}} & \rightarrow & 0.
\end{array}$$

where $(T_N\mathrm{Div}(X))^*$ denotes the dual of the group of T_N-invariant divisors $T_N\mathrm{Div}(X)$ and $A^i(\Delta)$ denotes the homogeneous part of degree $i, 1 \leq i \leq r$, of the Chow ring $A(\Delta)$. In this case, we have

$$\begin{array}{ll} T_N\mathrm{Div}(X) & = \bigoplus_{\rho\in\Delta(1)} \mathbf{Z}V(\rho) \\ A^1(\Delta) & = \sum_{\rho\in\Delta(1)} \mathbf{Z}v(\rho) \\ A^{r-1}(\Delta) & = \sum_{\tau\in\Delta(r-1)} \mathbf{Z}v(\tau), \end{array}$$

where $V(\rho)$ is the closures of the codimension-one T_N-orbit $\mathrm{orb}(\rho)$ corresponding to each cone $\rho \in \Delta(1)$, and $v(\rho)$ becomes the linearly equivalent class of $V(\rho)$. For any $\tau \in \Delta(r-1)$, $v(\tau)$ is defined similarly.

If Δ is assumed to be simplicial, each $\sigma \in \Delta(k)$ can be expressed as $\sigma = \rho_1 + \cdots + \rho_k$ for distinct $\rho_1, ..., \rho_k \in \Delta(1)$. In this case, we denote by $v(\sigma) := v(\rho_1)v(\rho_2)\cdots v(\rho_k)$. Then we have the following.

Proposition 4.1. (Danilov [3], Park [14]). *Let Δ be a simplicial fan for $N \cong \mathbf{Z}^r$. Then we have*

$$A^k(\Delta) = \sum_{\sigma\in\Delta(k)} \mathbf{Q}v(\sigma) \qquad \textit{for any} \qquad 0 \leq k \leq r.$$

As we have seen above, the structure and the number of cones in the fan tell us some informations of the properties of toric varieties.

5. Number of faces

¿From now on, we investigate the number of faces of fans or corresponding polytopes.

If $T_N\mathrm{emb}(\Delta)$ is a projective compact toric variety, then the number of cones in Δ is dual to the number of faces of the corresponding polytope $P := \Box_h$, as we have seen in Theorem 2.1. So, we may consider the number of faces of $P = \Box_h$, instead of the number cones in Δ. Let f_j be the number of j-dimensional faces of P and $h_k := \dim_{\mathbf{Q}} A^k(\Delta)$. Then we obtain

$$h_k = \sum_{j=0}^{k} (-1)^{k-j} \binom{r-j}{r-k} f_{j-1} \qquad \text{for} \qquad 0 \leq k \leq r$$

and

$$f_{j-1} = \sum_{k=0}^{j} \binom{r-k}{r-j} h_k \qquad \text{for} \qquad 0 \leq j \leq r.$$

If P is simplicial, then The *Dehn-Sommerville equalities* $h_k = h_{r-k}$, $0 \leq k \leq r$, hold (cf. [10] and [15]).

The Dehn-Sommerville equlities tell us more about the number of faces than Poincaré equation does. But from the generalized Dehn-Sommerville equations, we get more informations. In this case, we need the definition of the flag f-vectors of polytopes.

Definition For an r-dimensional convex polytope P, and for a subset $S \subset \{0, 1, \cdots, r-1\}$, we denote by f_S the number of chains of faces in P,

$$F_1 \subset F_2 \subset \cdots \subset F_k, \quad \text{with} \quad S = \{\dim F_1, \cdots, \dim F_k\}.$$

In this case, the chain $F_1 \subset F_2 \subset \cdots \subset F_k$ is calld the *S-flag.* The vector consisting of all the numbers f_S, $S \subset \{0, 1, \cdots, r-1\}$, is called the *flag f-vector* of P.

For the precise definition of flag f-vectors, see [1] and [7].

Proposition 5.1. (Billera and Björner [1], Liu [7]). *Let P be a convex polytope, S any subset of $\{0, 1, \cdots, r-1\}$, and $f = (f_S)$ the flag f-vector of P.*

(1) We have

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S\cup\{j\}} \quad = \quad (1-(-1)^{k-i-1}) f_S,$$

where $i, k \in S \cup \{-1, d\}, i \leq k-2$, and $S \cap \{i+1, \cdots, k-1\} = \emptyset$.

(2) The flag f-vectors of zonotopes satisfy the above (1), which is called the generalized Dehn-Sommerville equtions*, and no other linear relations not implied by these.*

Let $\mathcal{H} = \{H_1, H_2, \cdots, H_s\}$ be a hyperplanes arrangement in $N_{\mathbf{R}}$ such that $H_1 \cap H_2 \cap \cdots \cap H_s = \{0\}$ and each H_i is defined over $\mathbf{Q}$. Then we obtain a fan $\Delta_{\mathcal{H}}$. The flag f-vectors of $\Delta_{\mathcal{H}}$ is dual to that of its corresponding zonotope Z, in the sense that $f_S(\Delta_{\mathcal{H}}) = f_{r-S}(Z)$, where $S = \{i_1, \cdots, i_k\} \subset \{1, \cdots, r\}$ and $r - S = \{r - i_k, \cdots, r - i_1\}$.

Theorem 5.2. (Liu [7]). *For a hyperplane arrangement $\mathcal{H}$ in $N_{\mathbf{R}}$ and $S = \{i_1, \cdots, i_k\} \subset \{1, \cdots, r\}$ with $k \geq 2$, we have*

$$\frac{f_S(\Delta_{\mathcal{H}})}{f_{i_k}(\Delta_{\mathcal{H}})} < \binom{i_k}{i_1, i_2 - i_1, \cdots, i_k - i_{k-1}} 2^{i_k - i_1}.$$

¿From these equations, we get more informations about the number of faces of polytopes. For example, we get the following.

Proposition 5.3. (Billera and Björner [1]). *Let P be a polytope and f_i be the number of the faces in P.*

(1) P be a 3-polytope if and only if $f_0 \leq 2f_2 - 4$ and $f_2 \leq 2f_0 - 4$.

(2) If Z is a 3-zonotope, then f_0, f_1 must be even integers, $f_0 \leq 2f_2 - 4$ and $f_2 \leq f_0 - 2$.

There are another tools, called *cd-index*, to get the number of faces of polytopes. ¿From these tools, we get more informations about the cone decompositions, and also the structure of Chow rings.

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