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THE FACTORIZATION THEOREM OF FREE ARRANGEMENTS

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ABSTRACT. We will survey the proof of *The Factorization Theorem* of H. Terao. If $\mathcal{A}$ is a free arrangement, the characteristic polynomial of the arrangement factors. The intersection lattice of $\mathcal{A}$, $L(\mathcal{A})$, is used to define the characteristic polynomial. $L(\mathcal{A})$ is partially ordered by reverse inclusion. We will also study The Order Complex and The Folkman Complex, which plays a role in the proof of the Factorization Theorem. We will follow Orlik and Terao's book [25].

1. Introduction

Recently, Bruce E. Sagan wrote a paper, "WHY THE CHARACTERISTIC POLYNOMIAL FACTORS" in the Bulletin of the American Mathematical Society (1999). He introduced the Möbius function, μ, of a partially ordered set which is a far-reaching generalization of the one in number theory. And the characteristic polynomial, χ, is defined as the generating function for μ. He surveyed three methods for proving that the characteristic polynomoial of a finite ranked lattice factors over the nonnegative integers. The first technique uses geometric ideas and is based on Zaslavsky's theory of signed graphs. The second approach is algebraic and employs results of Saito and Terao about free hyperplane arrangements. The third approach is based on a purely combinatorial theorem of Stanley about supersolvable lattices and its generalizations. In this paper we will survey the second approach more in detail. Sagan explained the Factorization Theorem for the class $\mathcal{IF}$ of inductively free arrangements and for the class $\mathcal{RF}$ of recursively free arrangements. We will explain the proof of the Factorization Theorem of free Arrangements in general.

2. Set Up

Let K be a field. Let V be an ℓ dimensional vector space over K. Let $S = K[x_1, \cdots, x_\ell]$ be the polynomial algebra. Let $F = K(x_1, x_2, \cdots, x_\ell)$ be the field of rational functions on V. A hyperplane H in V is a codimension one subspace of V. An arrangement $\mathcal{A}$ in V is a finite set of hyperplanes in V.

Each hyperplane H in V has a defining form

$$\alpha_H = a_1x_1 + a_2x_2 + \cdots + a_\ell x_\ell \ (a_i \in K)$$

unique up to a constant multiple.

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A defining polynomial for $\mathcal{A}$ is given by

$$Q = Q(\mathcal{A}) = \prod_{H\in\mathcal{A}} \alpha_H.$$

The module of derivaions, $Der_K(S)$, consists of all K-linear maps $\theta : S \rightarrow S$ satisfying

$$\theta\ (\ fg\ )\ =\ f\theta(g) + g\theta(f).$$

The module of $\mathcal{A}$-derivations is defined by

$$D(\mathcal{A})\ =\ \{\theta \in Der_K(S) \mid \alpha_H | \theta(\alpha_H)\ for\ all\ H \in \mathcal{A}\ \}.$$

We say that the Arrangement $\mathcal{A}$ is free if $D(\mathcal{A})$ is a free S-module.

We set

$$\Omega^p(V) = \bigoplus_{1\leq i_1<\cdots<i_p\leq\ell} F(dx_{i_1} \wedge \cdots \wedge dx_{i_p}).$$

We agree that $\Omega^0(V) = F$. We call $\Omega^p(V)$ the module of rational differential p-forms on V.

We define a map $d : \Omega^p(V) \rightarrow \Omega^{p+1}(V)$ as follows: For $f \in F$, we define

$$df = \sum_{k=1}^{\ell} \frac{\partial f}{\partial x_k} dx_k.$$

For $\omega = \sum f_{i_1\cdots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, where $1 \leq i_1 < \cdots < i_p \leq \ell$ and $f_{i_1\cdots \imath_p} \in F$, we define

$$dw = \sum_{k=1}^{\ell} \sum (\partial f_{i_1\cdots i_p}/\partial x_k) dx_k \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

The map d is called exterior differentiation.

Let

$$\Omega^p[V] = \bigoplus_{1\leq i_1<\cdots<i_p\leq\ell} S(dx_{i_1} \wedge \cdots \wedge dx_{i_p}).$$

We agree that $\Omega^0[V] = S$. The elements of $\Omega^p[V]$ are called regular differential p-forms on V.

We define the module of logarithmic forms of $\mathcal{A}$ by

$$\Omega^p(\mathcal{A}) = \{\omega \in \Omega^p(V) \mid Q\omega \in \Omega^p[V]\ and\ Q(d\omega) \in \Omega^{p+1}[V]\}.$$

3. Möbius function and Characteristic polynomial

Let $L = L(\mathcal{A})$ be the set of all subspaces of V which are intersections of elements of $\mathcal{A}$. We call $L(\mathcal{A})$ the intersection lattice of $\mathcal{A}$. A partial order will be given to $L(\mathcal{A})$ by reverse inclusion. We define $\mathcal{A}_X$ as the set of hyperplanes in $\mathcal{A}$ which contains X. L_X denotes $L(A_X)$. We may define the Möbius function as follows:

$$\begin{aligned} \mu(X,X) &= 1 \\ \sum_{X\leq Z\leq Y} \mu(X,Z) &= 0 \end{aligned}$$

if $X, Y\ \in L(\mathcal{A})$ and $X < Y$.

We define $\mu(X)\ =\ \mu(V,X)$.

The Characteristic polynomial of $\mathcal{A}$ is defined by

$$\chi(\mathcal{A},t)\ =\ \sum_{X\in L(\mathcal{A})} \mu(X) t^{\dim X}.$$

The next result is the Möbius inversion formula.

Proposition 3.1. *([25]) Let $f,\ g$ be functions on $L(\mathcal{A})$ with values in an abelian group. Then*

$$g(Y) \;=\; \sum_{X\;\in\;L_Y} f(X)$$

if and only if

$$f(Y) \;=\; \sum_{X\;\in\;L_Y} \mu(X,Y)g(X).$$

4. Lattice Homology

A generalization of the Folkman complex called the Order complex with functors.

Definition 4.1. Let P be a partially ordered set. Let $K \;=\; K(P)$ be the simplicial complex associated to P as follows:
(1) the vertices of K are the elements of P,
(2) a set of vertices $\{X_0, \cdots, X_q\}$ spans a q-simplex if and only if it is a linearly ordered subset of P; after relabeling, $X_0 \;<\; \cdots \;< X_q$.

Given a poset P and the associated simplicial complex $K(P)$, let $K(P)$ be the corresponding geometric complex called the Order complex.

Definition 4.2. Let $\mathcal{A}$ be an Arrangement. Let $K(L \setminus \{V, T\})$ be the simplicial complex associated to the poset obtained from $L(\mathcal{A})$ by deleting its minimal and maximal elements. The Folkman complex is the corresponding geometric complex.

Definition 4.3. A covariant functor $F \;:\; L(\mathcal{A}) \;\to\; (S - \mathrm{Mod})$ is called *local* if the localization at P of the map

$$\nu_{X(P),X} \;:\; F(X(P)) \;\to\; F(X)$$

is an isomorphism for all $P \;\in\; \mathrm{Spec}(S)$ and for all $X \;\in\; L(\mathcal{A})$.

Let q be a nonnegative integer. Then, the covariant functor
$F \;:\; L(\mathcal{A}) \;\to\; (S - \mathrm{Mod})$ by $F(X) \;=\; \Omega^q(\mathcal{A}_X)$ is a *local functor.*
This proves the following theorem.

Theorem 4.4. *([25]) Let $X \;\in\; L(\mathcal{A})$ and let p be a nonnegative integer. Then*

$$\sum_{Y \in L_x} \mu(Y,X)\mathrm{Poin}(\Omega^p(\mathcal{A}_y), x)$$

has a pole at $x \;=\; 1$of order at most $\dim X$.

5. The Factorization Theorem of free Arrangements

If $\mathcal{A}$ is a free Arrangement, then $D(\mathcal{A})$ has a homogeneous basis $\theta_1, \cdots, \theta_\ell$ and the degree set $\{d_1, \cdots, d_\ell\} = \{\mathrm{deg}\theta_1, \cdots, \mathrm{deg}\theta_\ell\}$ depends only on $\mathcal{A}$. We call the degree set (allowing multiplicites),$exp(\mathcal{A})$.

If an Arrangement is free, $\Omega^1(\mathcal{A})$ is free. And $\Omega^1(\mathcal{A})$ has a homogeneous basis $\omega_1,\ \omega_2,\ \cdots,\ \omega_\ell$ with $\mathrm{pdeg}(\omega_i) \;=\; -b_i$ for all i.

Theorem 5.1. *([25]) If $\Omega^1(\mathcal{A})$ is a free S-modul with basis $\omega_1, \cdots, \omega_\ell$, then $\Omega^p(\mathcal{A})$ is free with basis $\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \mid 1 \leq i_1 < \cdots < i_p \leq \ell\}$.*

The above theorem implies that

$$\mathrm{Poin}(\Omega^p(\mathcal{A}),x) \;=\; \sum x^{-b_{i_1}\;\cdots\;-b_{i_p}}/(1-x)^\ell$$

where the sum is over the set $\{(i_1,\cdots,i_p) \mid 1 \;\leq\; i_1 \;<\; \cdots \;<\; i_p \;\leq\; \ell\}$.

Let

$$\Psi(\mathcal{A};x,t) \;=\; \sum_{p=0}^{\ell}\mathrm{Poin}(\Omega^p(\mathcal{A}),x)(t(1-x))^p.$$

Let y be an indeterminate. Then

$$\sum_{p=0}^{\ell}\mathrm{Poin}(\Omega^p(\mathcal{A}),x)y^p \;=\; \prod_{i=1}^{\ell}((1+x^{-b_i}y)/(1-x)).$$

Now set $y \;=\; t(1-x)-1$ and divide by $1-x$ in each factor. Then we get the following theorem.

Theorem 5.2. *([25]) If $\mathcal{A}$ is a free arrangement with exp$(\mathcal{A}) \;=\; \{b_1,\cdots,b_\ell\}$, then*

$$\Psi(\mathcal{A};x,t) \;=\; \prod_{i=1}^{\ell}(tx^{-b_i}-(x^{-1}+x^{-2}+\cdots+x^{-b_i})).$$

A priori $\Psi(\mathcal{A};x,t)$ may have a pole at $x \;=\; 1$ because each $\Omega^p(\mathcal{A})$ is a finite S-module. But it is proved that $\Psi(\mathcal{A};x,t)$ has no pole at$x \;=\; 1$. Also $\chi(\mathcal{A},t) \;=\; \Psi(\mathcal{A};1,t)$ is proved by use of Theorem 4.4.

This proves the Factorization Theorem of free Arrangements.

Theorem 5.3. *([25]) If $\mathcal{A}$ is a free Arrangement with exp$(\mathcal{A}) \;=\; \{d_1,d_2,\cdots,d_\ell\}$, then*

$$\chi(\mathcal{A},t) \;=\; (t-d_1)(t-d_2)\cdots(t-d_\ell).$$

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