Trends in Mathematics
Information Center for Mathematical Sciences
Volume 2, Number 1, September 1999, Pages 1–

FRITZ JOHN'S CONVEXITY THEOREM AND DISCRETE DYNAMICS

MAU-HSIANG SHIH

Abstract. We consider the result of great impact by Fritz John in his work on Convex Geometry. John showed in 1948 that the boundary of any convex region centrally symmetric with respect to a point in $\mathbf{R}^n$ lies between two concentric homothetic ellipsoids of ratio $1/\sqrt{n}$. This result has become very important for Geometric Algorithms. The purpose of this talk is to discuss its matrix-theoretic version and its new application to Discrete Dynamics. The content is mainly taken from papers: T. Ando and M.-H. Shih, SIAM J. Matrix Anal (1998); M.-H. Shih, Linear Algebra Appl. (1999).

1. Fritz John's convexity theorem

Fritz John's in 1948 [5] showed that the boundary of any convex region centrally symmetric in $\mathbf{R}^n$ lies between two concentric homothetic ellipsoids of ratio $1/\sqrt{n}$. (The constant $1/\sqrt{n}$ is optimal.) This basic result has become very important in Geometric Algorithms, see Grötshel, Lovasz and Schrijver [4]. John's proof was based on Lagrange's multipliers rule where the subsidiary conditions are inequalities.

2. Matrix-theoretic version

Let us recall that a set E in $\mathbf{C}^n$ is an **ellipsoid** if there exist a vector $a \in \mathbf{C}^n$ and a positive definite matrix Q such that

$$E \;\equiv\; E(Q,a) \;=\; \{\; x \in C^n \;;\; \langle Q(x-a), x-a\rangle \leq 1 \;\}.$$

Theorem 1. *Let $\{\, A_\lambda ; \lambda \in \Lambda \,\}$ be a bounded set of $n \times n$ positive semidefinite matrices such that*

$$\sup_{\lambda\in\Lambda} \langle A_\lambda x, x\rangle > 0 \quad \textit{for all } x \in \mathbf{C}^n \textit{with}\, x \neq 0.$$

Then there is an $n \times n$ positive definite matrix H_o such that

$$\langle H_o x, x\rangle \;\geq\; \sup_{\lambda\in\Lambda}\langle A_\lambda x, x\rangle \;\geq\; \frac{1}{n}\langle H_o x, x\rangle \quad \textit{for all}\, x \in \mathbf{C}^n.$$

The proof of Theorem 1 is based on the following construction.

Lemma. *Let A be an $n \times n$ complex matrix with $I \geq A \geq 0$ and let $0 \leq \alpha < \frac{1}{n}$. If for some unit vector e_1*

$$\langle Ae_1, e_1\rangle \leq \alpha,$$

then, with the rank projection $P_1 = e_1 \otimes e_1*$, *the matrix*

$$T = n\alpha P_1 + \frac{n(1-\alpha)}{(n-1)}(I - P_1)$$

satisfies

$$T \geq A \quad \textit{and} \det(T) < 1.$$

To see that Theorem 1 is a matrix-theoretic version of complex John's convexity theorem, let us mention the following:

Representation of a norm: *For any norm* $|||\cdot|||$ *on* $\mathbf{C}^n$ *there is a family of semidefinite matrices* $\{A_\lambda; \lambda \in \Lambda\}$ *such that*

$$|||x|||^2 = \sup_{\lambda\in\Lambda}\langle A_\lambda x, x\rangle \quad \textit{for all}\; x \in \mathbf{C}^n.$$

Now, let us state the uniqueness H_o of Theorem 1.

Proposition. *Consider the set of* $n \times n$ *positive definite matrices*

$$M = \{A > 0 \quad ; \quad \langle Ax, x\rangle \geq |||x|||^2 \;,\; x \in \mathbf{C}^n\}.$$

Then $H_o \in M$ *with minimum determinant is uniquely determined.*

The proof of proposition is based on the following:

(i) (Ky Fan) : On the set of $n \times n$ positive definite matrices the map

$$A \;\mapsto\; \det(A)^{\frac{1}{n}}$$

is **strictly concave.**

(ii) For $B, C > 0$,

$$\langle\, (\, B^{-1} + C^{-1}\,)^{-1}x, x\,\rangle \;=\; \inf\{\,\langle\, By, y\,\rangle + \langle\, C(x-y), x-y\,\rangle\;;\; y \in \mathbf{C}^n\}.$$

3. Nonautonomous Lyapunov inequalities

For an $n \times n$ complex matrix A, the inequalities

$$A^*HA \;<\; H$$

is called "Lyapunov inequalities" in the control literature. Since the existence of positive definite H such that $A^*HA \;<\; H$ is equivalent to the existence of positive definite P such that

$$PA + A^*P \;<\; 0\,.$$

The following is a discussion about nonautonomous Lyapunov inequalities.

For an $n \times n$ complex matrix A, $r(A)$ stands for the spectral radius of A and $\|A\|$ for the operator norm of A associated with the $\mathbf{C}^n$ norm $\|x\|$. Let C be the set of $n \times n$ complex matrices. For $m = 1, 2, \ldots,$ C^m is the set of all products of matrices in C of length m. Denote by $\hat{r}(C)$ the **joint spectral radius** of C[6], that is,

$$\hat{r}(C) \;=\; \lim_{m\to\infty}\sup\,[\sup_{A\in C^m}\;\|A\|]^{\frac{1}{m}}.$$

Denote by $r(C)$ the **generalized spectral radius** of C[2], that is,

$$r(C) \;=\; \lim_{m\to\infty}\sup\,[\sup_{A\in C^m}\;r(A)]^{\frac{1}{m}}.$$

It is clear that $r(C) \leq \hat{r}(C)$. Daubechies and Lagarias [2] conjectured that $r(C) = \hat{r}(C)$ (ie. generalized Gelfand spectral radius formula) if C consists of finitely amny $n \times n$ real matrices. Berger and Wong [1] proved this conjecture even if C is bounded; their method of proof was based on tools from ring theory. Recently, however, Elsner [3] gave an analytic - geometric proof and Shih [7] gave an analytic - combinational proof.

Theorem 2. **Let** C **be a bounded set of** $n \times n$ **complex matrices. If** $\hat{r}(C) < \frac{1}{\sqrt{n}}$**, then there is a positive definite matrices** H **and** $0 < \gamma < 1$ **such that**

$$A^* H A \leq \gamma H \quad for\, all\, A \in C.$$

The constant $\frac{1}{\sqrt{n}}$ **is optimal**.

The existence of H is based on Theorem 1. The optimality is based on the generalized Gelfand spectral radius formula.

References

1. M. A. Berger and Y. Wang, Bounded semigroups of matrices, Linear Algebra Appl. 166(1992), 21-27.

2. I. Daubechies and J. C. Lagarias, Sets of matrices all infinite products of which converge, Linear Algebra Appl. 161(1992), 227-263.

3. L. Elsner, The generalized spectral radius theorem : An analytic - geometric proof, Linear Algebra Appl. 220(1995), 151-159.

4. M. Grötschel, L. Lovasz and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization*, Springer - Verlag, 1988.

5. F. John, Extremum problems with inequalities as subsidiary conditions, in Studies and Essays, presented to R. Courant on his 60th birthday, Intercience, New York, 1948.

6. G.-C. Rota and G. Strang, A note on the joint spectral radius, Indag. Math. 22(1960), 379-381.

7. M.-H. Shih, Simultaneous Schur stability, Linear Algebra Appl. 287(1999), 323-336.

Department of Mathematics, Chung Yuan Chrisitian University, Chung-Li, Taiwan