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LYAPUNOV EXPONENTS FOR PRODUCTS OF RANDOM MATRICES

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Abstract. For each product of random matrices there associates a Lyapunov exponent. To evaluate the Lyapunov exponent one needs to find the invariant measure on the projective space $P(\mathbf{R}^d)$. The goal of this article is to study the invariant measure and to calculate the Lyapunov exponent explicitly.

1. Introduction

Let $\{Y_n, n \in \mathbf{N}\}$ be a sequence of i.i.d. random $d \times d$ invertible matrices with common distribution μ.

We assume that μ has support in $SL(d, \mathbf{R})$, set of real $d \times d$ matrices with determinant one and that $\mathbf{E}[\log^+ \|Y_1\|] < \infty$, where $\log^+ x = \max\{\log x, 0\}$. Let $S_n = Y_n \cdots Y_1$. Suppose that a usual vector norm and a usual matrix norm in $\mathbf{R}^d$ have been chosen.

Definition 1. *The Lyapunov exponent γ associated with μ is defined by*

$$\gamma = \lim_{n \to \infty} \frac{1}{n} \mathbf{E}[\log \|Y_n \cdots Y_1\|].$$

The Lyapunov exponent gives a measure of the exponential growth (decay) rate of the matrix norm.

The existence of the Lyapunov exponent can be easily proved by considering the subadditive sequence $\mathbf{E}[\log \|Y_n \cdots Y_1\|]$. Some explicit results are the following.

Theorem 1. *([9]) Let Y_1 be upper triangular. Then*

$$\gamma = \max_{1 \le i \le d} \mathbf{E}(\ln |(Y_1)_{ii}|).$$

Theorem 2. *([8]) If the d^2 entries of A_1 are independent Gaussian variables with mean zero and variance σ^2. Then*

$$\gamma = \ln \sigma + \frac{1}{2}[\ln 2 + \Psi(d/2)],$$

where Ψ is the digamma function, $\Psi(x) = \Gamma'(x)/\Gamma(x)$ and Γ is the standard gamma function.

It is proved by Furstenberg and Kesten [4] that under some irreducible condition on the random matrices and if $\mathbf{x}$ is a unit vector independent on $\{Y_n\}$, then with probability one,

$$\gamma = \lim_{n \to \infty} \frac{1}{n} \log \|Y_n \cdots Y_1 \mathbf{x}\|.$$

However, in most of the cases, the Lyapunov exponent cannot be calculated directly from μ. The formula for γ involves an auxiliary measure on the projective space $P(\mathbf{R}^d)$ [3].

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Let $\mathbf{x}$ be a unit vector in $\mathbf{R}^d$ and $\mathbf{u}_1 = \mathbf{x}$. For $n = 2, 3, \cdots$, let

$$\mathbf{u}_n = \frac{Y_{n-1} \cdots Y_1 \mathbf{x}}{\|Y_{n-1} \cdots Y_1 \mathbf{x}\|}.$$

It is clear that the process $\{(Y_n, \mathbf{u}_n), n \geq 1\}$ is a Markov chain on the phase space $SL(d, \mathbf{R}) \times S^{d-1}$, and that

$$\log \|Y_n \cdots Y_1 \mathbf{x}\| = \sum_{k=1}^{n} \log \|Y_k \mathbf{u}_k\|.$$

Thus if the process is ergodic then one expects that the limit

$$\beta(\mathbf{x}) = \lim_{n \to \infty} \frac{1}{n} \log \|Y_n \cdots Y_1 \mathbf{x}\|$$

exists almost surely and can be expressed as an average with respect to an invariant measure on the phase space. This leads to the consideration of the invariant measure on the projective space $P(\mathbf{R}^d)$ of $\mathbf{R}^d$.

For two non-zero vectors $\mathbf{x}, \mathbf{y} \in \mathbf{R}^d$ we say $\mathbf{x} \sim \mathbf{y}$ if $\mathbf{x} = c\mathbf{y}$ for some $c \in \mathbf{R}$. The projective space $P(\mathbf{R}^d)$ is the quotient space $\mathbf{R}^d \backslash \{\mathbf{0}\} / \sim$. For $\mathbf{x} \in \mathbf{R}^d \backslash \{\mathbf{0}\}$, let $\overline{\mathbf{x}}$ denote its equivalence class in $P(\mathbf{R}^d)$. For $M \in SL(d, \mathbf{R})$ we set $M \cdot \overline{\mathbf{x}} = \overline{M\mathbf{x}}$. Let μ be a probability measure on $SL(d, \mathbf{R})$, and ν a probability measure on $P(\mathbf{R}^d)$.

Definition 2. *The probability measure $\mu * \nu$ on $P(\mathbf{R}^d)$ is a probability measure on $P(\mathbf{R}^d)$ which satisfies*

$$\int f(\overline{\mathbf{x}}) d\mu * \nu(\overline{\mathbf{x}}) = \int \int f(M \cdot \overline{\mathbf{x}}) d\mu(M) d\nu(\overline{\mathbf{x}})$$

for every bounded Borel function f on $P(\mathbf{R}^d)$. We say that ν is μ-invariant if $\mu * \nu = \nu$.

The following theorem [5] gives the relationship between the invariant measures and the Lyapunov exponent.

Theorem 3. *Let $\{Y_n, n \in \mathbf{N}\}$ be a sequence of i.i.d. random matrices with common distribution μ. Suppose that μ has support in $SL(d, \mathbf{R})$ and that $\mathbf{E}[\log^+ \|Y_1\|] < \infty$. Then with probability one,*

$$\gamma = \sup \int \int \log \frac{\|M\mathbf{x}\|}{\|\mathbf{x}\|} d\mu(M) d\nu(\overline{\mathbf{x}})$$

where the sup *is taken over all μ-invariant measures.*

In the case that there is only one invariant measure on $P(\mathbf{R}^d)$ we have a simple expression for γ. The difficulty of calculating the Lyapunov exponent comes from finding the invariant measure. One explicit example is the following [7].

Theorem 4. *With*

$$Y_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we suppose that $(a, b)^T$ and $(c, d)^T$ are i.i.d. $N(0, \Sigma)$ random vectors. Then the unique invariant measure is Cauchy $(0, 1)$ and

$$\gamma = -\frac{1}{2}\hat{\gamma} + \frac{1}{2} \ln[\frac{1}{2} tr(\Sigma) + \sqrt{det(\Sigma)}],$$

where $\hat{\gamma}$ is the Euler constant: $\hat{\gamma} = 0.577215669 \ldots$.

2. The Existence and Uniqueness

The existence of the invariant measures on $P(\mathbf{R}^d)$ is a simple consequence of the fact that $P(\mathbf{R}^d)$ being compact and separable. Let ν_n be a sequence of probability measures on $P(\mathbf{R}^d)$. We say that ν_n converges weakly to a probability measure ν on $P(\mathbf{R}^d)$ if

$$\lim_{n\to\infty}\int_{P(\mathbf{R}^d)} f(\overline{\mathbf{x}})\,d\nu_n(\overline{\mathbf{x}}) = \int_{P(\mathbf{R}^d)} f(\overline{\mathbf{x}})\,d\nu(\overline{\mathbf{x}})$$

for every bounded continuous function f on $P(\mathbf{R}^d)$. Let μ^k denote the k-fold convolution of μ.

Theorem 5. *([3]) There exists at least one μ-invariant measure on $P(\mathbf{R}^d)$.*

Proof :
Let ν_0 be any probability measure on $P(\mathbf{R}^d)$, and

$$\nu_n = \frac{1}{n}\sum_{k=1}^{n}\mu^k * \nu_0.$$

Since $P(\mathbf{R}^d)$ is separable and compact there exists a weakly convergent subsequence of $\{\nu_n\}$ and its limit, say ν, is a probability measure on $P(\mathbf{R}^d)$. For any n,

$$\begin{aligned}
\mu * \nu_n &= \frac{1}{n}\sum_{k=1}^{n}(\mu^{k+1} * \nu_0)\\
&= \frac{1}{n}\sum_{k=1}^{n}(\mu^{k} * \nu_0) + \frac{1}{n}(\mu^{n+1} * \nu_0 - \mu * \nu_0)\\
&= \nu_n + \frac{1}{n}(\mu^{n+1} * \nu_0 - \mu * \nu_0)
\end{aligned}$$

so that, letting $n \to \infty$ along a subsequence, $\mu * \nu = \nu$.

In the sequel we will consider the special case $d = 2$. Let T_μ (resp. G_μ) be the smallest closed semigroup (resp. group) containing the support of μ. A probability measure ν on $P(\mathbf{R}^2)$ is said to be continuous if $\nu(\overline{\mathbf{x}}) = 0$ for every $\overline{\mathbf{x}} \in P(\mathbf{R}^2)$.

Given a subset H of $SL(2,\mathbf{R})$, we say that H is contracting if there exists a sequence $\{M_n, n \geq 1\}$ in H for which $\|M_n\|^{-1}M_n$ converge to a rank one matrix. We say that H is strongly irreducible if there does not exist a subset V of $\mathbf{R}^2$ which is a finite union of one-dimensional subspaces of $\mathbf{R}^2$ such that $M(V) = V$ for every $M \in H$. It is not hard to see that a subgroup H of $SL(2,\mathbf{R})$ being contracting is equivalent to being non-compact.

Theorem 6. *([3]) If G_μ is strongly irreducible and contracting. Then there exists a unique μ-invariant measure and which is continuous.*

If G_μ is not contracting (i.e., G_μ being compact) then all matrices in G_μ must have norm one. Therefore the Lyapunov exponent vanishes. If G_μ is contracting and strongly irreducible Theorem 6 gives that there is only one μ-invariant measure. We may ask what if G_μ is contracting but not strongly irreducible, can there be two or more invariant measures? The answer is positive.

Example :
Let μ be concentrated on

$$A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix},$$

and $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$ be the corresponding directions of $(1,0)^T$, $(0,1)^T$. It is clear that G_μ is contracting but not strongly irreducible. Let $\nu(\overline{\mathbf{x}}_1) = p$ and $\nu(\overline{\mathbf{x}}_2) = 1-p$, $0 < p < 1$. We have for each bounded Borel function f on $P(\mathbf{R}^2)$,

$$\begin{aligned}\mu * \nu(f) &= \int\int f(M\cdot\overline{\mathbf{x}})\,d\mu(M)\,d\nu(\overline{\mathbf{x}})\\ &= p\int f(M\cdot\overline{\mathbf{x}}_1)\,d\mu(M) + (1-p)\int f(M\cdot\overline{\mathbf{x}}_2)\,d\mu(M)\\ &= pf(A\cdot\overline{\mathbf{x}}_1) + (1-p)f(A\cdot\overline{\mathbf{x}}_2)\\ &= pf(\overline{\mathbf{x}}_1) + (1-p)f(\overline{\mathbf{x}}_2)\\ &= \nu(f).\end{aligned}$$

This shows that ν is an invariant measure.

However, in this case, G_μ being contracting but not strongly irreducible, we have a simple formula for the Lyapunov exponent. We now assume that all the matrices in G_μ cannot be diagonalized simultaneously. For the situation that G_μ contains diagonal matrices only, explicit result for the Lyapunov exponent has been given [9].

Theorem 7. *If G_μ is contracting but not strongly irreducible, then there exist vectors $\mathbf{x}_1$, $\mathbf{x}_2 \in \mathbf{R}^2$ such that*

$$\gamma = \frac{1}{2}\int \left[\log\frac{\|M\mathbf{x}_1\|}{\|\mathbf{x}_1\|} + \log\frac{\|M\mathbf{x}_2\|}{\|\mathbf{x}_2\|}\right] d\mu(M).$$

Proof :

By Proposition 4.3 ([2], p.31) we have that there exist $\overline{\mathbf{x}}, \overline{\mathbf{x}}_1, \overline{\mathbf{x}}_2 \in P(\mathbf{R}^2)$ such that $B = \{M\cdot\overline{\mathbf{x}} | M \in G_\mu\} = \{\overline{\mathbf{x}}_1, \overline{\mathbf{x}}_2\}$. Suppose that $\overline{\mathbf{x}}_1 \neq \overline{\mathbf{x}}_2$. Since $I \in G_\mu$, $\overline{\mathbf{x}} \in B$. We claim that $\nu(\{\overline{\mathbf{x}}_1\}) = \nu(\{\overline{\mathbf{x}}_2\}) = \frac{1}{2}$. Let $f = 1_{B^c}$. We have

$$\begin{aligned}\nu(B^c) &= \int f(\overline{\mathbf{x}})\,d\nu(\overline{\mathbf{x}})\\ &= \int\int f(M\cdot\overline{\mathbf{x}})\,d\mu(M)\,d\nu(\overline{\mathbf{x}})\\ &= \int [f(\overline{\mathbf{x}}_1)\mu(\{M|M\cdot\overline{\mathbf{x}} = \overline{\mathbf{x}}_1\}) + f(\overline{\mathbf{x}}_2)\mu(\{M|M\cdot\overline{\mathbf{x}} = \overline{\mathbf{x}}_2\})]\,d\nu(\overline{\mathbf{x}})\\ &= 0.\end{aligned}$$

This shows that $\nu(\{\overline{\mathbf{x}}_1\}) + \nu(\{\overline{\mathbf{x}}_2\}) = 1$. Now let $f = 1_{\{\overline{\mathbf{x}}_1\}}$. We have

$$\begin{aligned}\nu(\{\overline{\mathbf{x}}_1\}) &= \int f(\overline{\mathbf{x}})\,d\nu(\overline{\mathbf{x}})\\ &= \int\int f(M\cdot\overline{\mathbf{x}})\,d\mu(M)\,d\nu(\overline{\mathbf{x}})\\ &= \int \nu(\{M^{-1}\cdot\overline{\mathbf{x}}_1\})\,d\mu(M)\\ &= \nu(\{\overline{\mathbf{x}}_1\})\mu(\{M|M^{-1}\cdot\overline{\mathbf{x}}_1 = \overline{\mathbf{x}}_1\}) + \nu(\{\overline{\mathbf{x}}_2\})\mu(\{M|M^{-1}\cdot\overline{\mathbf{x}}_1 = \overline{\mathbf{x}}_2\}).\end{aligned}$$

Together with $\mu(\{M|M^{-1}\cdot\overline{\mathbf{x}}_1 = \overline{\mathbf{x}}_1\}) + \mu(\{M|M^{-1}\cdot\overline{\mathbf{x}}_1 = \overline{\mathbf{x}}_2\}) = 1$ ($\mu(\{M|M^{-1}\cdot\overline{\mathbf{x}}_1 = \overline{\mathbf{x}}_2\}) > 0$, otherwise G_μ, after suitable change of basis, will contain only diagonal matrices) and $\nu(\{\overline{\mathbf{x}}_1\}) + \nu(\{\overline{\mathbf{x}}_2\}) = 1$ we solve the last equation and get $\nu(\{\overline{\mathbf{x}}_1\}) = \nu(\{\overline{\mathbf{x}}_2\}) = \frac{1}{2}$. If $\mathbf{x}_1$, $\mathbf{x}_2$ are vectors in directions $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$, then

$$\gamma = \frac{1}{2}\int \left[\log\frac{\|M\mathbf{x}_1\|}{\|\mathbf{x}_1\|} + \log\frac{\|M\mathbf{x}_2\|}{\|\mathbf{x}_2\|}\right] d\mu(M).$$

This formula holds also for $\overline{\mathbf{x}}_1 = \overline{\mathbf{x}}_2$.

In the case G_μ being strongly irreducible and contracting we know that there is only one μ-invariant measure. In the next section we give another example for explicit calculation of the invariant measure and the Lyapunov exponent. It needs further effort to find the invariant measures in more cases.

3. The random Schrödinger equation

The one-dimensional Anderson model [1] is the simplest quantum-mechanical approximation to the scattering of electrons in crystals with impurities. The model is given by the Hamiltonian operator

$$\mathbf{H}u_n = -\Delta u_n + V(n)u_n$$

acting in the Hilbert space $l^2(\mathbf{Z})$. Here Δ is the discrete Laplacian

$$\Delta u_n = u_{n+1} + u_{n-1} - 2u_n,$$

and $V(n)$, the random potentials, are independent identically distributed (i.i.d.) real random variables with common distribution μ. Here, we use the same notation for distributions of random matrices and random potentials. We are interested in the eigenvalue equation (*random Schrödinger equation*)

$$\mathbf{H}u_n = Eu_n,$$

i.e.,

$$2u_n - u_{n-1} - u_{n+1} + V(n)u_n = Eu_n. \tag{1}$$

We can rewrite equation (1) in matrix form as

$$\begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix} = \begin{bmatrix} V(n)+2-E & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix}. \tag{2}$$

Let

$$\mathbf{x}_n = \begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix},$$

and

$$Y_n = \begin{bmatrix} V(n)+2-E & -1 \\ 1 & 0 \end{bmatrix}.$$

We may write (2) in the following form

$$\mathbf{x}_{n+1} = Y_n \mathbf{x}_n.$$

Thus, if the initial value $\mathbf{x}_0$ is given, we can integrate the solution by a product of random matrices

$$\mathbf{x}_{n+1} = Y_n \cdots Y_1 \mathbf{x}_0.$$

Note that Y_n and their products belong to the special linear group $SL(2, \mathbf{R})$. If we define

$$a_n = \frac{u_n}{u_{n-1}} \in \dot{\mathbf{R}} = \mathbf{R} \cup \{\infty\},$$

then

$$a_{n+1} = (V(n)+2-E) - \frac{1}{a_n}.$$

It is clear that $(V(n), a_n)$ is a Markov chain. A μ-invariant measure ν on $\dot{\mathbf{R}}$ is a measure satisfying

$$\int\int f(v+2-E-y^{-1})d\nu(y)d\mu(v) = \int f(y)d\nu(y)$$

for every bounded Borel function f on $\dot{\mathbf{R}}$.

Let $f(t) = \int e^{-itv}\, d\mu(v)$ be the characteristic function of μ. Klein and Speiss [6] show the following theorem on the regularity of the invariant measure.

Theorem 8. *Let $n \in \mathbf{N}$, $n \geq 3$, and f be n times differentiable on $(0, \infty)$ with bounded derivatives such that*

$$\lim_{t \to \infty} f^{(k)}(t) = 0$$

for all $k = 0, \ldots, n$. Then the unique μ-invariant measure ν is absolutely continuous and $d\nu/dx$ is of class $C^{[(n-1)/2]-1}$.

Thus, suppose that μ satisfies the assumption of Theorem 8, then the integral equation for the density ϕ of ν will be

$$\phi(\overline{\mathbf{x}}) = \int \phi\left(\frac{1}{v + 2 - E - \overline{\mathbf{x}}}\right) \frac{1}{(v + 2 - E - \overline{\mathbf{x}})^2}\, d\mu(v). \tag{3}$$

Our result for the random Schrödinger equation is the following [10].

Theorem 9. *Suppose that the distribution of the random potential is Cauchy $(E - 2, \beta)$. Then*

$$\psi(\overline{\mathbf{x}}) = \frac{\alpha}{\pi(\overline{\mathbf{x}}^2 + \alpha^2)}$$

with $\alpha > 0$ and $\alpha^2 - \alpha\beta - 1 = 0$ satisfies equation (3), i.e., the invariant measure is Cauchy $(0, \alpha)$. Furthermore, the Lyapunov exponent is $\log \alpha$.

Proof. We need to show that

$$\psi(\overline{\mathbf{x}}) = \int \psi\left(\frac{1}{v - \overline{\mathbf{x}}}\right) \frac{1}{(v - \overline{\mathbf{x}})^2} \frac{\beta}{\pi(v^2 + \beta^2)}\, dv. \tag{4}$$

Indeed,

$$\begin{aligned}
&\int \frac{\alpha}{\pi[(v - \overline{\mathbf{x}})^{-2} + \alpha^2]} \frac{1}{(v - \overline{\mathbf{x}})^2} \frac{\beta}{\pi(v^2 + \beta^2)}\, dv \\
&= \frac{\alpha\beta}{\pi^2} \int \frac{1}{\alpha^2 (v - \overline{\mathbf{x}})^2 + 1} \frac{1}{(v^2 + \beta^2)}\, dv \\
&= \frac{\alpha\beta}{\pi^2} \int \left[\frac{av + b}{\alpha^2 (v - \overline{\mathbf{x}})^2 + 1} + \frac{cv + d}{(v^2 + \beta^2)}\right] dv \\
&= \frac{\alpha\beta}{\pi}\left[\frac{a\overline{\mathbf{x}} + b}{\alpha} + \frac{d}{\beta}\right],
\end{aligned} \tag{5}$$

where where

$$\begin{aligned}
a &= \frac{-2\alpha^4 \overline{\mathbf{x}}}{\alpha^4 \overline{\mathbf{x}}^4 + 2\alpha^2(\alpha^2\beta^2 + 1)\overline{\mathbf{x}}^2 + (\alpha^2\beta^2 - 1)^2} \\
b &= \frac{3\alpha^4 \overline{\mathbf{x}}^2 + \alpha^4\beta^2 - \alpha^2}{\alpha^4 \overline{\mathbf{x}}^4 + 2\alpha^2(\alpha^2\beta^2 + 1)\overline{\mathbf{x}}^2 + (\alpha^2\beta^2 - 1)^2} \\
c &= \frac{2\alpha^2 \overline{\mathbf{x}}}{\alpha^4 \overline{\mathbf{x}}^4 + 2\alpha^2(\alpha^2\beta^2 + 1)\overline{\mathbf{x}}^2 + (\alpha^2\beta^2 - 1)^2} \\
d &= \frac{\alpha^2 \overline{\mathbf{x}}^2 - \alpha^2\beta^2 + 1}{\alpha^4 \overline{\mathbf{x}}^4 + 2\alpha^2(\alpha^2\beta^2 + 1)\overline{\mathbf{x}}^2 + (\alpha^2\beta^2 - 1)^2}.
\end{aligned}$$

Now, by substituting a, b, c and d into (5), we obtain

$$\int \psi\left(\frac{1}{v - \overline{\mathbf{x}}}\right) \frac{1}{(v - \overline{\mathbf{x}})^2} \frac{\beta}{\pi(v^2 + \beta^2)}\, dv = \frac{1}{\pi} \frac{\alpha(\alpha\beta + 1)}{\alpha^2 \overline{\mathbf{x}}^2 + (\alpha\beta + 1)^2}.$$

Equation (4) follows by $\alpha^2 = \alpha\beta + 1$.

The formula for the Lyapunov exponent is

$$\gamma = \int \int \log \frac{\|M\mathbf{x}\|}{\|\mathbf{x}\|} d\mu(M)\, d\nu(\overline{\mathbf{x}}),$$

where

$$M\mathbf{x} = \begin{bmatrix} v & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} vx - y \\ x \end{bmatrix}.$$

Let $\overline{\mathbf{x}} = x/y$. Then $M \cdot \overline{\mathbf{x}} = \overline{M\overline{\mathbf{x}}} = (v\overline{\mathbf{x}} - 1)\overline{\mathbf{x}}^{-1}$. Again, we use the same notation μ for the distributions of random matrices and random potential. We get

$$\gamma = \frac{1}{2} \int \int \log \frac{(v\overline{\mathbf{x}} - 1)^2 + \overline{\mathbf{x}}^2}{\overline{\mathbf{x}}^2 + 1} d\mu(v) d\nu(\overline{\mathbf{x}}).$$

By straightforward calculation and using the fact $\mu*\nu(f) = \nu(f)$ for $f(\overline{\mathbf{x}}) = \log(\overline{\mathbf{x}}^2 + 1)$, we end up with

$$\gamma = \log \alpha.$$

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