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SEMILINEAR ELLIPTIC EQUATIONS IN UNBOUNDED DOMAINS

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ABSTRACT. We give a survey on two important problems in seminear elliptic equations

1 Two Important Problems

For the existence of solutions of semilinear elliptic equations, there are two most difficult problems that lack $(PS)-$conditions: the first problem is the critical exponential problems

$$\text{(CE)} \qquad \begin{cases} -\Delta u = u^{(N+2)/(N-2)} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N > 2$; the second problem is the unbounded domain problems

$$\text{(UD)} \qquad \begin{cases} -\Delta u + u = u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $2 < p < 2^*$, $2^* = \frac{2N}{N-2}$ for $N > 2$, $2^* = \infty$ for $N = 2$, and Ω is an unbounded domain in $\mathbb{R}^N$. The existence of solutions of the two important problems which has been the focus of a great deal of research in recent years is affected by the shape of the domain Ω and has parallel results.

2 Critical Exponential Problems

Theorem 1. *Suppose that Ω is a star-shaped domain, then there does not exist any solution of the CE problem.*

Note that a ball is a star-shaped domain, so there does not exist any solution of the CE problem. However,

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Theorem 2. *Suppose that Ω is an annular domain, then there exists a radial solution of CE problem.*

Nirenberg Conjecture: Let Ω be a ball with a hole which is not a non-annular, does there exist any solution of the CE problem in Ω?

Let $\Omega = \mathbb{R}^N$ then there are infinite many positive solutions

$$u_{\epsilon,z_0}(z) = \frac{[N(N-2)\epsilon^2]^{\frac{N-2}{4}}}{[\epsilon^2 + |z-z_0|^2]^{(N-2)/2}}, \ \ \epsilon > 0, \ z_0 \in \mathbb{R}^N$$

of the CE problem, with the same energy

$$\int_{\mathbb{R}^N} |\nabla u_{\epsilon,z_0}(z)|^2 dx = c < \infty, \ \ \text{for } \epsilon > 0, z_0 \in \mathbb{R}^N.$$

Theorem 3. *If u is a positive solution such that $\int_{\mathbb{R}^N} |\nabla u|^2 < \infty$; or $u(z) = o(|z|^{2-N})$; or u is radial symmetry, then u is of form in part 1.*

Using the existence and the uniqueness (every solution is a ground state solution) of solutions, Coron asserted the Nirenberg conjecture partially.

Theorem 4. *Suppose that Ω is a ball. If there exist $z_0 \in \mathbb{R}^N$, $0 < R_1 < R_2$ such that $\frac{R_2}{R_1}$ is sufficiently large and*

$$\{z \in \mathbb{R}^N \mid R_1 \le |z - z_0| \le R_2\} \subset \Omega,$$
$$\{z \in \mathbb{R}^N \mid |z - z_0| \le R_1\} \not\subseteq \Omega,$$

then there exists a solution of the CE problem.

Theorem 5. *There exists $R_0 \in (0,1)$. For $R \in (0, R_0)$, there exists $r_0(R) > 0$ such that if $r \in (0, r_0)$ then there exists a solution of the CE problem in $\Omega = A_R \setminus \mathbb{S}^r_{0,1}$, where $A_R = \{z \in \mathbb{R}^N \mid 0 < R < |z| < 1\}$ and $\mathbb{S}^r_{0,1} = \{(x,y) \in \mathbb{S}^r \mid 0 < y < 1\}$.*

3 Unbounded Domain Problems

Let for $r > 0, \ R > 0, \ \ a < b$,

$$\begin{aligned}
&A_R = \{z \in \mathbb{R}^N \mid 0 < R < |z| < 1\};\\
&\mathbb{R}^N(z_0; R) = \{z \in \mathbb{R}^N \mid |z - z_0| < R\};\\
&\mathbb{S}^r = \{(x,y) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid |x| < r\};\\
&\mathbb{S}^r_{a,b} = \{(x,y) \in \mathbb{S} \mid a < y < b\};\\
&\mathbb{S}^r_a = \{(x,y) \in \mathbb{S}^r \mid a < y\};\\
&D^r_s = \mathbb{S}^r_0 \cup B^N(0; s);\\
&E^r_s = \mathbb{R}^N \setminus \overline{D^r_s}.
\end{aligned}$$

where A_R is an annular, $\mathbb{B}^N(z_0; R)$is a ball, $\mathbb{S}^r$ is the strip domain in $\mathbb{R}^N$, $\mathbb{S}^r_a$ is the upper half strip domain, D^r_s is an interior flask domain, and E^r_s is an exterior flask domain.

Definition 6. *A proper unbounded domain Ω in $\mathbb{R}^N$ is an Esteban-Lions domain if there is $\chi \in \mathbb{R}^N$, $||\chi|| = 1$ such that $n(z) \cdot \chi \geq 0$, and $n(z) \cdot \chi \not\equiv 0$ on $\partial\Omega$, where $n(z)$ denotes the unit outward normal to $\partial\Omega$ at the point z.*

Example 7. *A typical example of Esteban-Lions domains is the upper half strip domain $\mathbb{S}_0^r$.*

Theorem 8. *There does not exist any solutions $H_0^1(\Omega)$ for the UD problem in an the upper half strip domain $\mathbb{S}_0^r$.*

The existence of solutions is affected by the shape of the domain Ω; but it is extremely difficult to see which domain Ω is solvable or unsolvable. Thus it is natural to start from the study of perturbations of Esteban-Lions domains.

Berestycki Conjecture: Let Ω be the upper half strip domain $\mathbb{S}_0^r$ with a hole, there exists a solution of the UD problem in Ω.

Notation:

$$a(u) = ||u||_{H^1}^2 = \int_\Omega \left(|\nabla u|^2 + u^2\right),$$

$$b(u) = ||u||_p^p = \int_\Omega |u|^p,$$

$$J(u) = \frac{1}{2}a(u) - \frac{1}{p}b(u).$$

$$I = \inf\{a(u) \mid b(u) = 1\},$$

$$\alpha_I = (\frac{1}{2} - \frac{1}{p})I^{p/(p-2)},$$

$$\mathbf{M} = \left\{u \in H_0^1(\Omega) \setminus \{0\} \mid a(u) = b(u)\right\},$$

$$\alpha_M = \inf_{v \in \mathbf{M}} J(v),$$

$$\Gamma = \{v \in C([0,1], H_0^1(\Omega)) \mid v(0) = 0, v(1) = e\}, \text{ where } J(e) = 0,$$

$$\alpha_\Gamma = \inf_{v \in \Gamma} \max_{t \in [0,1]} J(v(t)).$$

Definition 9.

1. *For $\beta \in \mathbb{R}$, a sequence $\{u_n\} \subset H_0^1(\Omega)$ is a $(PS)_\beta$-sequence for J if $J(u_n) \to \beta$ and $J'(u_n) \to 0$ strongly as $n \to \infty$;*
2. *$\beta \in \mathbb{R}$ is a $(PS)_\beta$-value for J if there is a $(PS)_\beta$-sequence for J;*
3. *J satisfies the $(PS)_\beta$-condition if every $(PS)_\beta$-sequence for J contains a convergent subsequence;*
4. *J satisfies the (PS)-condition if, for every $\beta \in \mathbb{R}$, every $(PS)_\beta$-sequence for J contains a convergent subsequence.*

Lemma 10. *α_I, α_M, and α_Γ are positive (PS)-values for J.*

Lemma 11. *Let $\{u_n\} \subset H_0^1(\Omega)$ be a $(PS)_\beta$-sequence for J with $\beta > 0$. Then there is a sequence $\{s_n\}$ in $\mathbb{R}^+$ such that $\{s_n u_n\} \subset \mathbf{M}$ and $J(s_n u_n) = \beta + o(1)$.*

Lemma 12. *Let $\{u_n\} \subset H_0^1(\Omega)$ be a $(PS)_\beta$-sequence for J with $\beta > 0$. Then*

1. $\beta \geq \alpha_I$; 2. $\beta \geq \alpha_M$; 3. $\beta \geq \alpha_\Gamma$.

Theorem 13. $\alpha_I = \alpha_M = \alpha_\Gamma$.

Rabinowitz [Minimax Methods in Critical Point Theory, 1984, p.19] asked if $\alpha_\Gamma = \alpha_{\Gamma'}$, where

$$\Gamma' = \{K \subset H_0^1(\Omega) \mid K \text{ is closed, connected, and } 0,\ e \in K\}$$

and $\alpha_{\Gamma'} = \inf_{K\in\Gamma'} \max_{u\in K} J(u)$. We answer his question affirmatively:

Corollary 14. $\alpha_\Gamma = \alpha_{\Gamma'}$.

Now $\alpha_I = \alpha_M = \alpha_\Gamma = \alpha_{\Gamma'}$.

Definition 15.

1. *We call that $\alpha(\Omega) = \alpha_M$ is the index of the energy functional J in Ω;*
2. *We call that a solution u of equation (UD) is a ground state solution if $J(u) = \alpha(\Omega)$, and is a higher energy solution if $J(u) > \alpha(\Omega)$.*

In order to study the Berestycki conjecture: there is a solution of the UD equation in the upper half strip domain $\mathbb{S}_0^r$ with a hole, we should understand not only the existence of the solution in its limiting domain (the strip domain $\mathbb{S}^r$) but also the uniqueness (every solution is a ground state solution) of the solutions in its limiting domain.

Theorem 16. *There exists a ground state solution of the UD problem in the strip domain $\mathbb{S}^r$.*

In order to study the symmetry of solutions in the strip domain $\mathbb{S}^r$, we assert the following two asymptotic behavior results:

Theorem 17. *(Basic) Let $u(x,y)$ be a C^2 solution of the UD problem in the strip domain $\mathbb{S}^r$, then*

$$\lim_{y\to\infty} u(x,y) = 0 \text{ uniformly in } x \in \mathbf{B}^{N-1}(0;r).$$

Theorem 18. *(Advanced) Let $u(x,y)$ be a C^2 solution of the UD problem in the strip domain $\mathbb{S}^r$. Then for any $0 < \delta < 1 + \lambda_1$ there exist $\alpha > 0$ and $\beta > 0$ such that*

$$\alpha\phi_1(x)e^{-\sqrt{1+\lambda_1+\delta}\,|y|} \le u(z) \le \beta\phi_1(x)e^{-\sqrt{1+\lambda_1-\delta}\,|y|},$$

for $z = (x,y) \in \mathbb{S}^r$, where λ_1 and $\phi_1(x)$ is the eigenpair of the Dirichlet problem in $\mathbf{B}^{N-1}(0;r)$.

Theorem 19. *Let $u(x,y)$ be a C^2 solution of the UD problem in the strip domain $\mathbb{S}^r$. Then u is radially symmetric in x and axially symmetric in y ; that is,*

$$u(x, y-\sigma) = u(|x|, |y-\sigma|) \text{for some } \sigma.$$

Uniqueness:

Theorem 20. *If $N = 1$, then the solution of the UD Equation is unique.*

For $N > 1$ it is still open.

Using Theorems 15 and 16, We assert partially the Berestycki conjecture:

Theorem 21. *Suppose that the solution of the UD problem in the infinite strip $\mathbb{S}^r$ is unique up to y-translations (or every solution is a ground state solution). Let Ω be the upper half strip $\mathbb{S}_0^r$ with a hole D. If there exists $h_0 > 0$ such that $D \subset \mathbb{S}_h^r$, then there is a positive higher energy solution v of the UD problem in Ω such that $\alpha(\mathbb{S}^r) < F(v) < 2^{\frac{p-2}{p}}\alpha(\mathbb{S}^r)$, where*

$$F(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + v^2 - \frac{1}{p}\int_\Omega |v|^p.$$

For $k = 1, 2, \cdots$, define $\Omega_k = \mathbb{S}_0^r \setminus \overline{B^N((0,h); \frac{1}{k})}$, where $h \geq 2h_0$, $\frac{1}{k} < d$. By Theorem 13, we have, for each k, a positive solution $u_k \in H_0^1(\Omega_k)$ of $-\Delta u_k + u_k = u_k^{p-1}$ in Ω_k satisfying

$$\alpha(\mathbb{S}^r) < F(u_k) < 2^{\frac{p-2}{p}}\alpha(\mathbb{S}^r)\,.$$

Moreover, we have the following dynamic systems of solutions $\{u_k\}$:

Theorem 22. $|\nabla u_k|^2 dz = c\delta_0 + o(1)$ *for some positive number c.*

It is natural to ask if Ω be the upper half strip $\mathbb{S}_0^r$ with k hole D_i, $i = 1, 2, \cdots, k$, then there are k solutions. We don't know the answer. If Ω be the entire space $\mathbb{R}^N$ with k hole $D_i, i = 1, 2, \cdots, k$. two articles by Italian mathematicians have shown the results, one use analysis and the other use algebraic topology. But both of them have gaps. However if we perturb the UD equation by a small function:

$$\text{(UDf)}\qquad \begin{cases} -\Delta u + u = u^{p-1} + f(z) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u \in H_0^1(\Omega), \end{cases}$$

then we have the multiplicity.

Theorem 23. *Let $D \subset\subset \mathbb{S}^r$ be a bounded $C^{1,1}$ domain, $f \in L^2(\Omega)$, $f \geq 0$, $f \neq 0, ||f||_{L^2}^2 \leq c(p)$ and there exist $\delta > 0$, $d > 0$ such that*

$$0 \leq f(z) \leq d\exp(-\sqrt{1+\lambda_1+\delta}|y|),\ \textit{for } z = (x,y) \in \Omega.$$

Then there are at least two positive solutions of the UD problem in $\Omega = \mathbb{S}^r \setminus D$.

There are two ways to perturb a Esteban-Lions domain Ω: one way is to make a hole from Ω; the other is to add another domain in it as follows:

Let $\Omega_0 = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2$ is bounded, $\alpha_i = \alpha(\Omega_i)$ for $i = 0, 1, 2$.

Theorem 24. *J satisfies the $(PS)_{\alpha_0}$-condition if and only if the inequality $\alpha_0 < \min\{\alpha_1, \alpha_2\}$ holds. In particular, if the inequality $\alpha_0 < \min\{\alpha_1, \alpha_2\}$ holds, then there is a ground state solution u of equation (1) in Ω_0.*

Theorem 25. *There exists $s_0 > 0$ such that the index $\alpha(D_s^r)$ admits a ground state solution if $s > s_0$, but $\alpha(D_s^r)$ does not admit any ground state solution if $s < s_0$.*

We have the following results for the exterior flask domains E_s^r :

Theorem 26. *There exist $r_1 > 0$ and $s_1 > 0$ such that if $r < r_1$ and $r < s < s_1$, then there is a positive solution u in $H_0^1(E_s^r)$ of the UD problem in the exterior flask domain E_s^r , where $N \geq 4$.*

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